**Math 221 Spring 2025 (Darij Grinberg): midterm 1**
due date: Sunday 2025-05-04 at 11:59PM on gradescope ( `https://www.gradescope.com/courses/1011749` ).
Please solve only **3 of the 6 exercises**.
**NO collaboration allowed – this is a midterm!**
(But you can still ask me questions.)

Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

**Exercise 1.** Let $n \in \mathbb{N}$, and let $q$ be any number. Prove that

$$(q-1)^2 \cdot \sum_{k=1}^{n} kq^{k-1} = nq^{n+1} - (n+1)\,q^n + 1.$$

**Exercise 2.** Let $(a_0, a_1, a_2, \ldots)$ be a sequence of integers defined recursively by

$$a_0 = 2, \qquad a_1 = 1,$$
$$a_n = a_{n-1} + 6a_{n-2} \qquad \text{for all } n \geq 2.$$

Prove that $a_n = 3^n + (-2)^n$ for each $n \in \mathbb{N}$.

**Exercise 3.** Let $n \in \mathbb{N}$. Prove that

$$\underbrace{1 + 2 + \cdots + n}_{= \sum_{k=1}^{n} k} = \underbrace{n^2 - (n-1)^2 + (n-2)^2 - (n-3)^2 \pm \cdots + (-1)^{n-1} 1^2}_{= \sum_{k=1}^{n} (-1)^{n-k} k^2}.$$

**Exercise 4.** Let $n \in \mathbb{N}$. Prove that

$$\prod_{i=1}^{n} \binom{2i-1}{i} = \prod_{i=1}^{n} \binom{2n-i}{i}.$$

**Exercise 5.** Let $(a_0, a_1, a_2, \ldots)$ be a sequence of integers defined recursively by

$$a_n = 1 + a_0 a_1 \cdots a_{n-1} \qquad \text{for all } n \geq 0.$$

(In particular, $a_0 = 1 + \underbrace{a_0 a_1 \cdots a_{0-1}}_{=(\text{empty product})=1} = 1 + 1 = 2$.) Here are the first few entries of this sequence:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $a_n$ | 2 | 3 | 7 | 43 | 1807 | 3263443 | 10650056950807 |

(notice the astronomical growth!).

**(a)** Prove that

$$a_{n+1} = a_n^2 - a_n + 1 \qquad \text{for each } n \geq 0.$$

**(b)** Prove that

$$\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_{n-1}} = 1 - \frac{1}{a_n - 1} \qquad \text{for each } n \geq 0.$$

Now, recall the Tower of Hanoi puzzle (as discussed in §1.1), and let $m_n$ denote the # of moves needed to win (= solve this puzzle) with $n$ disks. As we have seen in Theorem 1.2.2, we have $m_n = 2^n - 1$ for each $n \in \mathbb{N}$.

Consider the variant of the Tower of Hanoi puzzle in which we have 4 instead of 3 pegs, but otherwise the rules of the game are the same (and the goal is still is to move all $n$ disks from peg 1 to peg 3). Let $t_n$ denote the # of moves needed to win this variant with $n$ disks.

**Exercise 6.** **(a)** Prove that

$$t_{a+b} \leq m_b + 2t_a \qquad \text{for any } a, b \in \mathbb{N}.$$

**(b)** Prove that $t_4 \leq 9$.