Math 221 Spring 2025 (Darij Grinberg): homework set 2
due date: Sunday 2025-04-20 at 11:59PM on gradescope (`https://www.gradescope.com/courses/1011749`).
Please solve only **3 of the 6 exercises**.

Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Exercise 1. **(a)** Prove that

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for each positive integer n.

(b) Find and prove a closed-form expression (i.e., no $\prod$ or $\sum$ signs) for

$$\prod_{i=2}^{n} \left(1 - \frac{1}{i}\right).$$

Exercise 2. Prove that

$$\prod_{i=1}^{n} \left(i! \cdot i^i\right) = n!^{n+1} \qquad \text{for each } n \in \mathbb{N}.$$

The *floor* $\lfloor x \rfloor$ of a real number x means the largest integer that is smaller or equal to x. For instance, $\lfloor 6.2 \rfloor = 6$ and $\lfloor 7.7 \rfloor = 7$ and $\lfloor 8 \rfloor = 8$. (In other words, $\lfloor x \rfloor$ is what you get if you round x down. Beware: $\lfloor -1.3 \rfloor$ is -2, not -1.)

Exercise 3. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=1}^{n} \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n+1}{2} \right\rfloor.$$

(In this exercise, you can freely use basic properties of even and odd numbers – such as Proposition 3.3.8 in the notes.)

Exercise 4. Let $(a_0, a_1, a_2, \ldots)$ be a sequence of integers defined recursively by

$$a_0 = 2, \qquad a_1 = 3,$$
$$a_n = 3a_{n-1} - 2a_{n-2} \qquad \text{for all } n \geq 2.$$

Prove that $a_n = 2^n + 1$ for each $n \in \mathbb{N}$.

Now, we recall again the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ that we got to know in §1.5. It is defined recursively by $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for each $n \geq 2$.

Exercise 5. **(a)** Let $k \in \mathbb{N}$. Show that

$$f_n^2 - f_{n+k} f_{n-k} = (-1)^{n-k} f_k^2 \qquad \text{for every integer } n \geq k.$$

(b) Which exercise from homework set #1 does this generalize?

Exercise 6. Prove that every $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfy

$$\sum_{k=0}^{n} \binom{n}{k} f_{m+k} = f_{m+2n}.$$