Math 221 Winter 2024 (Darij Grinberg): homework set 3
due date: Sunday 2024-02-04 at 11:59PM on gradescope (`https://www.gradescope.com/courses/684379`).
Please solve only **3 of the 6 exercises**.

Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Exercise 1. Let $a, b \in \mathbb{Z}$. Prove the following:
(a) If $a \mid b$, then $a^k \mid b^k$ for each $k \in \mathbb{N}$.
(b) Let $n \in \mathbb{Z}$. If $a \equiv b \mod n$, then $a^k \equiv b^k \mod n$ for each $k \in \mathbb{N}$.

Exercise 2. Let $n, d, a, b \in \mathbb{Z}$, and assume that $d \neq 0$ and $da \equiv db \mod dn$.
(a) Prove that $a \equiv b \mod n$.
(b) Show by an example that $a \equiv b \mod dn$ is not necessarily true (i.e., we cannot simply cancel the d from da and db while leaving the dn unchanged).

Exercise 3. Let n be any integer. Prove the following:
(a) If n is odd, then $8 \mid n^2 - 1$.
(b) If $3 \nmid n$, then $3 \mid n^2 - 1$.
[**Hint:** In part **(a)**, write n as $2k + 1$. In part **(b)**, write n as $q \cdot 3 + r$ and consider the possible values for r.]

Exercise 4. Let p be a positive integer.
Assume that you are given p-cent coins and $(p + 1)$-cent coins (each in infinite supply).
Prove that you can pay n cents using these coins for every integer $n \geq p^2 - p$.
In other words, prove that each integer $n \geq p^2 - p$ can be written as $a(p+1) + bp$ with $a, b \in \mathbb{N}$.

Now for two more properties of binomial coefficients:

Exercise 5. **(a)** Prove that $k\dbinom{n}{k} = n\dbinom{n-1}{k-1}$ for any two numbers n and k.
(b) Prove that $\sum\limits_{k=0}^{n} k\dbinom{n}{k} x^k = nx(x+1)^{n-1}$ for any positive integer n and any number x.
[**Hint:** Don't forget about cases like $k = 0$ and $k \notin \mathbb{N}$.]

Exercise 6. Let $(f_0, f_1, f_2, \ldots)$ be the Fibonacci sequence. Let $n \in \mathbb{N}$. Prove that

$$2^{n-1} \cdot f_n = \sum_{k=0}^{n} \binom{n}{2k+1} \cdot 5^k.$$

[**Hint:** The 5 on the right hand side looks suspiciously like the 5 in $\dfrac{1+\sqrt{5}}{2}$, whereas the binomial coefficients look like the binomial formula...]