The Peel exact sequence for hook Specht modules via exterior algebra

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This is a mostly expository note devoted to one of the first nontrivial results in the modular representation theory of the symmetric groups: Peel's hook exact sequence. This sequence has been introduced by Peel in [Peel71] and revisited by Künzer in [Kuenze15, Propositions 4.2.3 and 4.2.4]. The aim of this note is to reprove its most basic properties in maximum generality (over arbitrary commutative rings satisfying $n = 0$ when necessary, not just finite fields) and more or less conceptually (using basic homological algebra rather than ad-hoc computations involving Young tableaux).

It took me a while to figure it out, and I suspect that I ended up rediscovering known properties of Koszul complexes. Yet I could not easily locate any of it in the literature, so I have written up my proof in some reasonable level of detail (not approaching that of my lecture notes [Grinbe25], however, as this is somewhat more advanced material).

I should note that Peel's article [Peel71] goes significantly beyond constructing the exact sequence; the present note does not supersede it.

1. Introduction

Peel's hook exact sequence is an exact sequence consisting of Specht modules. The classical way to define them is in terms of Young tableaux. But we will use an equivalent definition using Vandermonde determinants, since it is easier and self-contained:

Let $\mathbf{k}$ be any commutative ring, and n a positive integer.

Consider the symmetric group S_n of the set $[n] = \{1, 2, \ldots, n\}$. It acts from the left on the polynomial ring $\mathcal{P}_n = \mathbf{k}[x_1, x_2, \ldots, x_n]$ by $\mathbf{k}$-algebra automorphisms that permute the variables ($\sigma \cdot x_i = x_{\sigma(i)}$ for all $\sigma \in S_n$ and $i \in [n]$). For any k elements $a_1, a_2, \ldots, a_k$ of any commutative ring, we let $V(a_1, a_2, \ldots, a_k)$ denote their Vandermonde determinant

$$V(a_1, a_2, \ldots, a_k) := \det\left(a_i^{j-1}\right)_{i,j\in[k]} = \prod_{1\leq i<j\leq k} \left(a_j - a_i\right).$$

Note that this is clearly an alternating function in the inputs $a_1, a_2, \ldots, a_k$.

For any $k \in [n]$, we let $\mathcal{S}^{\lambda^k}$ denote the $\mathbf{k}$-linear span of the Vandermonde determinants $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ in $\mathcal{P}_n$ (where $(i_1, i_2, \ldots, i_k)$ ranges over $[n]^k$). This is an S_n-subrepresentation of $\mathcal{P}_n$, and is well-known to be isomorphic to the Specht module of the hook partition $\lambda^k := \left(n-k+1, 1^{k-1}\right)$ (where "1^{k-1}" means $k-1$ many 1s in sequence). (The explicit isomorphism can be found, e.g., in [Grinbe25, Corollary 5.6 **(b)**]. Note that the Vandermonde determinant $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ is an alternating function in the inputs $i_1, i_2, \ldots, i_k$; thus, the $\mathbf{k}$-module $\mathcal{S}^{\lambda^k}$ is spanned by the $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ with $i_1, i_2, \ldots, i_k$ distinct, and even just by those with $i_1 < i_2 < \cdots < i_k$. Note that this is a particular case of how Specht defined Specht modules in the first place.) Peel (in [Peel71]) denotes $\mathcal{S}^{\lambda^k}$ as $S(k-1, n)$.

Note that by the standard basis theorem for Specht modules, the $\mathbf{k}$-module $\mathcal{S}^{\lambda^k}$ has a basis consisting of those $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ with $1 = i_1 < i_2 < \cdots < i_k$ (since these correspond to the standard Young tableaux of shape λ^k).

Now, assume that $n = 0$ in $\mathbf{k}$. Then, Peel (in [Peel71, §3]) and Künzer (in [Kuenze15, Proposition 4.2.3]) show that for each $k \in [n-1]$, there is an S_n-representation homomorphism

$$\begin{aligned} f_k : \mathcal{S}^{\lambda^k} &\to \mathcal{S}^{\lambda^{k+1}}, \\ V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right) &\mapsto \sum_{s\in[n]} V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_s\right). \end{aligned} \tag{1}$$

Note that the sum can just as well be restricted to the $s \in [n] \setminus \{i_1, i_2, \ldots, i_k\}$ only, since the addends for $s \in \{i_1, i_2, \ldots, i_k\}$ are 0. Furthermore, they show that these maps f_k form an exact sequence

$$0 \to \mathcal{S}^{\lambda^1} \overset{f_1}{\to} \mathcal{S}^{\lambda^2} \overset{f_2}{\to} \cdots \overset{f_{n-1}}{\to} \mathcal{S}^{\lambda^n} \to 0$$

that has a $\mathbf{k}$-linear (but not $\mathbf{k}[S_n]$-linear) chain contraction.

The purpose of this note is to prove this in a conceptual and readable way. The proof in Peel's [Peel71] is not fully clear to me, and only considers the case when $\mathbf{k}$ is a field. Künzer in [Kuenze15] only shows the existence of the maps f_k, not the exactness of the sequence; it is also computational and intransparent (though very elementary). Thus, I hope that this note has some use to others.

2. The proof

As I said, the proof I am giving uses just basic homological algebra (morphisms and chain contractions of complexes) and exterior powers. But we need to get some notation introduced and auxiliary results proved.

In the following, we do **not** assume that $n = 0$ in $\mathbf{k}$ unless we explicitly say so.

Let N be the free $\mathbf{k}$-module $\mathbf{k}^n$ with its standard basis $(e_1, e_2, \ldots, e_n)$. This is the natural representation of S_n, where S_n acts on $\mathbf{k}^n$ by permuting the basis ($\sigma \cdot e_i = e_{\sigma(i)}$ for each $\sigma \in S_n$ and $i \in [n]$).

Consider the S_n-invariant element

$$e := e_1 + e_2 + \cdots + e_n \in N. \tag{2}$$

Consider the exterior algebra $\Lambda N = \bigoplus_{i=0}^{n} \Lambda^i N$. It has a basis $(e_I)_{I \subseteq [n]}$, where we set

$$e_{\{i_1 < i_2 < \cdots < i_k\}} := e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \qquad \text{for each } \{i_1 < i_2 < \cdots < i_k\} \subseteq [n].$$

The exterior algebra ΛN as well as each exterior power $\Lambda^i N$ is an S_n-representation, equipped with the diagonal S_n-action:

$$g (v_1 \wedge v_2 \wedge \cdots \wedge v_i) = gv_1 \wedge gv_2 \wedge \cdots \wedge gv_i$$

for all $g \in S_n$ and $v_1, v_2, \ldots, v_i \in N$.

Let $\varepsilon : \Lambda N \to \Lambda N$ be the $\mathbf{k}$-linear map sending each w to $e \wedge w$.

Let $\varepsilon_1 : \Lambda N \to \Lambda N$ be the $\mathbf{k}$-linear map sending each w to $e_1 \wedge w$.

The exterior algebra ΛN is known to be a supercommutative superalgebra with $\mathbb{N}$-grading given by placing $\Lambda^i N$ in degree i. A *superderivation* of ΛN shall mean a $\mathbf{k}$-linear map $d : \Lambda N \to \Lambda N$ that satisfies the *super-Leibniz rule*

$$d(ab) = d(a) \cdot b + (-1)^i a \cdot d(b) \qquad \text{for all } i \in \mathbb{N} \text{ and } a \in \Lambda^i N \text{ and } b \in \Lambda N.$$

Let $\partial : \Lambda N \to \Lambda N$ be the $\mathbf{k}$-linear map defined by

$$\partial \left(e_{\{i_1 < i_2 < \cdots < i_k\}} \right) = \sum_{p=1}^{k} (-1)^{p-1} e_{\left\{i_1 < i_2 < \cdots < \widehat{i_p} < \cdots < i_k\right\}} \qquad \text{for each } \{i_1 < i_2 < \cdots < i_k\} \subseteq [n].$$

Here (and in the following), the "magician's hat" $\widehat{\ }$ is understood to vanish whatever stands under it; thus, "$i_1 < i_2 < \cdots < \widehat{i_p} < \cdots < i_k$" means "$i_1 < i_2 < \cdots < i_{p-1} < i_{p+1} < i_{p+2} < \cdots < i_k$".

Let $\partial_1 : \Lambda N \to \Lambda N$ be the $\mathbf{k}$-linear map defined by

$$\partial_1 \left(e_I\right) = \begin{cases} e_{I \setminus \{1\}}, & \text{if } 1 \in I \\ 0, & \text{if } 1 \notin I \end{cases} \qquad \text{for each } I \subseteq [n] .$$

We note that both maps ∂ and ∂_1 can be described in more canonical ways. Namely, if we let $(e_1^*, e_2^*, \ldots, e_n^*)$ denote the dual basis to the basis $(e_1, e_2, \ldots, e_n)$ of N (that is, each $e_i^* \in N^*$ is the $\mathbf{k}$-linear map $N \to \mathbf{k}$ that sends each vector to its i-th coordinate), and if we set $e^* := e_1^* + e_2^* + \cdots + e_n^* \in N^*$, then all $k \in \mathbb{N}$ and $v_1, v_2, \ldots, v_k \in N$ satisfy

$$\begin{aligned} &\partial \left(v_1 \wedge v_2 \wedge \cdots \wedge v_k\right) \\ &= \sum_{p=1}^{k} (-1)^{p-1} e^* \left(v_p\right) \cdot v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v_p} \wedge \cdots \wedge v_k \end{aligned} \tag{3}$$

and

$$\begin{aligned} &\partial_1 \left(v_1 \wedge v_2 \wedge \cdots \wedge v_k\right) \\ &= \sum_{p=1}^{k} (-1)^{p-1} e_1^* \left(v_p\right) \cdot v_1 \wedge v_2 \wedge \cdots \wedge \widehat{v_p} \wedge \cdots \wedge v_k. \end{aligned} \tag{4}$$

In other words, in terms of interior products (see, e.g., [Winitz23, §2.3.1]), we have $\partial = \iota_{e^*}$ and $\partial_1 = \iota_{e_1^*}$.

Both maps ∂ and ∂_1 are superderivations. The maps ε and ε_1 shift the degree by 1 upwards, while the maps ∂ and ∂_1 shift it by 1 downwards. Hence, their images and kernels are $\mathbb{N}$-graded $\mathbf{k}$-submodules of ΛN, and their cokernels inherit the $\mathbb{N}$-grading from ΛN. When we shall speak of $(\operatorname{Coker} \partial)_i$, we will mean the i-th graded component of this grading on $\operatorname{Coker} \partial$. Explicitly,

$$(\operatorname{Coker} \partial)_i = \left(\Lambda^i N\right) \diagup \partial \left(\Lambda^{i+1} N\right) \qquad \text{for each } i \in \mathbb{N}.$$

Note that the maps ε and ∂ are morphisms of S_n-representations, whereas ε_1 and ∂_1 are merely $\mathbf{k}$-module morphisms. Thus, $\operatorname{Coker} \partial$ and its graded components $(\operatorname{Coker} \partial)_i$ are S_n-representations.

The following is well-known:

Proposition 2.1. Each of these four maps $\varepsilon, \partial, \varepsilon_1, \partial_1$ makes the graded algebra ΛN into a long exact sequence: i.e., we have

$$\begin{aligned} \operatorname{Ker} \varepsilon &= \operatorname{Im} \varepsilon, & \operatorname{Ker} \partial &= \operatorname{Im} \partial, \\ \operatorname{Ker} \varepsilon_1 &= \operatorname{Im} \varepsilon_1, & \operatorname{Ker} \partial_1 &= \operatorname{Im} \partial_1. \end{aligned}$$

Moreover, these four exact sequences are each other's chain contractions: i.e., we have

$$\begin{aligned} \partial \varepsilon_1 + \varepsilon_1 \partial &= \operatorname{id} \qquad \text{and} \\ \partial_1 \varepsilon + \varepsilon \partial_1 &= \operatorname{id} . \end{aligned}$$

Proof. Clearly, $\varepsilon^2 = 0$ (since $\underbrace{e \wedge e}_{=0} \wedge w = 0$ for each $w \in \Lambda N$) and $\varepsilon_1^2 = 0$ (likewise). Very easily, $\partial_1^2 = 0$. By a standard computation from homological algebra, we have $\partial^2 = 0$ as well. (This can also be proved without getting one's hands dirty, by defining the symmetric bilinear form $\langle \cdot, \cdot \rangle$ on ΛN by $\langle e_I, e_J \rangle = \begin{cases} 1, & \text{if } I = J; \\ 0, & \text{if } I \neq J \end{cases}$ for all $I, J \subseteq [n]$. Indeed, this bilinear form $\langle \cdot, \cdot \rangle$ is nondegenerate and is easily seen to satisfy

$$\langle \varepsilon (e_I), e_J \rangle = \langle e_I, \partial (e_J) \rangle \qquad \text{for all } I, J \subseteq [n];$$

thus, the map ∂ is the adjoint of ε with respect to this bilinear form. Hence, $\partial^2 = 0$ follows from $\varepsilon^2 = 0$.)

Since ∂ is a superderivation, each $w \in \Lambda N$ satisfies $\partial (e_1 \wedge w) = \underbrace{\partial (e_1)}_{=1} w - e_1 \wedge \partial (w) = w - e_1 \wedge \partial (w)$, that is, $\partial (\varepsilon_1 (w)) = w - \varepsilon_1 (\partial (w))$, that is, $\partial (\varepsilon_1 (w)) + \varepsilon_1 (\partial (w)) = w$. Thus, $\partial \varepsilon_1 + \varepsilon_1 \partial = \operatorname{id}$ holds. Similarly, $\partial_1 \varepsilon + \varepsilon \partial_1 = \operatorname{id}$ holds (since $\partial_1 (e) = 1$).

The "Ker=Im" identities now follow easily: For instance, let us prove $\operatorname{Ker} \varepsilon = \operatorname{Im} \varepsilon$. The $\operatorname{Im} \varepsilon \subseteq \operatorname{Ker} \varepsilon$ inclusion follows from $\varepsilon^2 = 0$. To prove the converse inclusion, fix $w \in \operatorname{Ker} \varepsilon$; then, $\varepsilon (w) = 0$; but $\partial_1 \varepsilon + \varepsilon \partial_1 = \operatorname{id}$ yields $\partial_1 (\varepsilon (w)) + \varepsilon (\partial_1 (w)) = w$, whence $w = \partial_1 \underbrace{(\varepsilon (w))}_{=0} + \varepsilon (\partial_1 (w)) = \varepsilon (\partial_1 (w)) \in \operatorname{Im} \varepsilon$. So $\operatorname{Ker} \varepsilon \subseteq \operatorname{Im} \varepsilon$ is proved, and with it $\operatorname{Ker} \varepsilon = \operatorname{Im} \varepsilon$. Likewise, the other three "Ker=Im" identities can be shown. Altogether, the proof of Proposition 2.1 is complete. □

Proposition 2.2. We have

$$\partial_1 \partial = -\partial \partial_1. \tag{5}$$

Hence,

$$\partial_1 (\operatorname{Im} \partial) \subseteq \operatorname{Im} \partial. \tag{6}$$

Thus, the map $\partial_1 : \Lambda N \to \Lambda N$ descends to a $\mathbf{k}$-linear map $\partial_1' : \operatorname{Coker} \partial \to \operatorname{Coker} \partial$ such that the diagram

$$\begin{array}{ccc} \Lambda^k N & \xrightarrow{\partial_1} & \Lambda^{k-1} N \\ \downarrow & & \downarrow \\ (\operatorname{Coker} \partial)_k & \xrightarrow[\partial_1']{} & (\operatorname{Coker} \partial)_{k-1} \end{array} \tag{7}$$

(where the vertical arrows are graded parts of the canonical projection $\Lambda N \to \operatorname{Coker} \partial$) is commutative for each $k > 0$.

Proof. The equality $\partial_1 \partial = -\partial \partial_1$ is easy to check directly from the definitions of ∂ and ∂_1. Alternatively, we can prove it abstractly using the fact that the superderivations of a superalgebra form a Lie superalgebra, so that the supercommutator of two superderivations is again a superderivation. Indeed, this shows that $\partial_1 \partial + \partial \partial_1$ (being the supercommutator of ∂_1 and ∂) is a superderivation. Since it sends the generators $\Lambda^1 N$ of ΛN to 0, it thus is 0 everywhere (by the super-Leibniz rule). Either way, (5) is proved.

Of course, (6) follows immediately from (5).

The existence of ∂_1' for which the diagram (7) commutes is an immediate consequence of (6). □

So far, this all was true for any commutative ring $\mathbf{k}$.

Now, we note that $\partial(e) = n$. Hence, if $n = 0$ in $\mathbf{k}$, then we gain extra properties:

Proposition 2.3. Assume that $n = 0$ in $\mathbf{k}$. We have

$$\varepsilon \partial = -\partial \varepsilon. \tag{8}$$

Thus, up to the usual $(-1)^{\deg}$ sign twist, ε is a degree-shifting endomorphism of the complex $(\Lambda N, \partial)$, and vice versa.

We furthermore have

$$\varepsilon(\operatorname{Im} \partial) \subseteq \operatorname{Im} \partial. \tag{9}$$

Consequently, the map $\varepsilon : \Lambda N \to \Lambda N$ descends to a $\mathbf{k}$-linear endomorphism $\varepsilon' : \operatorname{Coker} \partial \to \operatorname{Coker} \partial$ of $\operatorname{Coker} \partial = (\Lambda N) / \operatorname{Im} \partial$ such that the diagram

$$\begin{array}{ccc} \Lambda^k N & \xrightarrow{\varepsilon} & \Lambda^{k+1} N \\ \downarrow & & \downarrow \\ (\operatorname{Coker} \partial)_k & \xrightarrow[\varepsilon']{} & (\operatorname{Coker} \partial)_{k+1} \end{array} \tag{10}$$

(where the vertical arrows are graded parts of the canonical projection $\Lambda N \to \operatorname{Coker} \partial$) is commutative for each $k \in \mathbb{N}$.

Proof. For each $w \in \Lambda N$, we have

$$\begin{aligned} \partial(\varepsilon(w)) &= \partial(e \wedge w) \qquad \text{(by the definition of } \varepsilon) \\ &= \underbrace{\partial(e)}_{=n=0} w - e \wedge \partial(w) \qquad \text{(since } \partial \text{ is a superderivation)} \\ &= -e \wedge \partial(w) = -\varepsilon(\partial(w)) \qquad \text{(by the definition of } \varepsilon). \end{aligned}$$

Hence, $\partial \varepsilon = -\varepsilon \partial$, so that $\varepsilon \partial = -\partial \varepsilon$. This proves (8). Thus, (9) immediately follows, and from it flows the existence of ε' that makes the diagram (10) commutative. ■ □

Theorem 2.4. Assume that $n = 0$ in $\mathbf{k}$. Then, the sequence

$$\cdots \xrightarrow{\varepsilon'} (\operatorname{Coker}\partial)_{k-1} \xrightarrow{\varepsilon'} (\operatorname{Coker}\partial)_{k} \xrightarrow{\varepsilon'} (\operatorname{Coker}\partial)_{k+1} \xrightarrow{\varepsilon'} \cdots$$

(where ε' was defined in Proposition 2.3) is exact and has a $\mathbf{k}$-linear chain contraction.

Proof. Recall the ∂_1' map from Proposition 2.2. Clearly, $(\varepsilon')^2 = 0$ because $\varepsilon^2 = 0$. Moreover, projecting the equality $\partial_1\varepsilon + \varepsilon\partial_1 = \operatorname{id}$ (from Proposition 2.1) onto $\operatorname{Coker}\partial$, we obtain $\partial_1'\varepsilon' + \varepsilon'\partial_1' = \operatorname{id}$, so that ∂_1' is a chain contraction for the sequence

$$\cdots \xrightarrow{\varepsilon'} (\operatorname{Coker}\partial)_{k-1} \xrightarrow{\varepsilon'} (\operatorname{Coker}\partial)_{k} \xrightarrow{\varepsilon'} (\operatorname{Coker}\partial)_{k+1} \xrightarrow{\varepsilon'} \cdots .$$

Thus, this sequence is exact, and Theorem 2.4 is proved. □

Now we come to the Peel modules $\mathcal{S}^{\lambda^k}$. As we recall, for each $k \in [n]$, the S_n-representation $\mathcal{S}^{\lambda^k}$ is defined as the $\mathbf{k}$-linear span of the Vandermonde determinants $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ in $\mathcal{P}_n$. We furthermore set

$$\mathcal{S}^{\lambda^k} := 0 \qquad \text{for all } k > n \text{ and also for } k = 0.$$

Note that this agrees with the original definition of $\mathcal{S}^{\lambda^k}$ as the span of the $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ for $k > n$ (since $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ is alternating as a function in $i_1, i_2, \ldots, i_k$), but not for $k = 0$. Yet it is the right way to define $\mathcal{S}^{\lambda^0}$, as we will see.

For each $k > 0$, we define the $\mathbf{k}$-linear map

$$\begin{aligned} \omega_k : \Lambda^k N &\to \mathcal{S}^{\lambda^k}, \\ e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} &\mapsto V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right). \end{aligned}$$

This map ω_k is well-defined, since $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ is an alternating function in its inputs $i_1, i_2, \ldots, i_k$. Moreover, ω_k is a morphism of S_n-representations (since $\sigma \cdot e_i = e_{\sigma(i)}$ and $\sigma \cdot x_i = x_{\sigma(i)}$ for all $\sigma \in S_n$ and $i \in [n]$).

We furthermore define $\omega_k : \Lambda^k N \to \mathcal{S}^{\lambda^k}$ to be the zero map 0 for $k > n$ and for $k = 0$.

We now claim the following:

Lemma 2.5. Let $k \in \mathbb{N}$. Then, $\omega_k \partial = 0$ on $\Lambda^{k+1} N$.

Proof. We WLOG assume that $k \in [n]$, since otherwise the claim is made obvious by the fact that $\omega_k = 0$.

The **k**-module $\Lambda^{k+1}N$ is spanned by the vectors $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k+1}}$ for all $(i_1, i_2, \ldots, i_{k+1}) \in [n]^{k+1}$. Hence, by linearity, it suffices to show that

$$\omega_k \left(\partial \left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k+1}}\right)\right) = 0 \qquad \text{for all } (i_1, i_2, \ldots, i_{k+1}) \in [n]^{k+1}.$$

Let us thus do this. Fix $(i_1, i_2, \ldots, i_{k+1}) \in [n]^{k+1}$. Then, the equality (3) shows that

$$\begin{aligned}
&\partial \left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k+1}}\right) \\
&= \sum_{p=1}^{k+1} (-1)^{p-1} \underbrace{e^* \left(e_{i_p}\right)}_{=1} \cdot e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_{k+1}} \\
&= \sum_{p=1}^{k+1} (-1)^{p-1} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_{k+1}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\omega_k \left(\partial \left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k+1}}\right)\right) \\
&= \omega_k \left(\sum_{p=1}^{k+1} (-1)^{p-1} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_{k+1}}\right) \\
&= \sum_{p=1}^{k+1} \underbrace{(-1)^{p-1}}_{=(-1)^{p+1}} \underbrace{\omega_k \left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge \widehat{e_{i_p}} \wedge \cdots \wedge e_{i_{k+1}}\right)}_{\substack{=V\left(x_{i_1}, x_{i_2}, \ldots, \widehat{x_{i_p}}, \ldots, x_{i_{k+1}}\right) \\ =\det \begin{pmatrix} x_{i_1}^0 & x_{i_2}^0 & \cdots & \widehat{x_{i_p}^0} & \cdots & x_{i_k}^0 & x_{i_{k+1}}^0 \\ x_{i_1}^1 & x_{i_2}^1 & \cdots & \widehat{x_{i_p}^1} & \cdots & x_{i_k}^1 & x_{i_{k+1}}^1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ x_{i_1}^{k-1} & x_{i_2}^{k-1} & \cdots & \widehat{x_{i_p}^{k-1}} & \cdots & x_{i_k}^{k-1} & x_{i_{k+1}}^{k-1} \end{pmatrix} \\ \text{(since } V(a_1, a_2, \ldots, a_k) = \det\left(a_j^{i-1}\right)_{i,j \in [k]} \text{ for all } a_1, a_2, \ldots, a_k)}} \\
&= \sum_{p=1}^{k+1} (-1)^{p+1} \det \begin{pmatrix} x_{i_1}^0 & x_{i_2}^0 & \cdots & \widehat{x_{i_p}^0} & \cdots & x_{i_k}^0 & x_{i_{k+1}}^0 \\ x_{i_1}^1 & x_{i_2}^1 & \cdots & \widehat{x_{i_p}^1} & \cdots & x_{i_k}^1 & x_{i_{k+1}}^1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ x_{i_1}^{k-1} & x_{i_2}^{k-1} & \cdots & \widehat{x_{i_p}^{k-1}} & \cdots & x_{i_k}^{k-1} & x_{i_{k+1}}^{k-1} \end{pmatrix}.
\end{aligned}$$

On the other hand, if we let A denote the $(k+1) \times (k+1)$-matrix

$$\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \\ x_{i_1}^0 & x_{i_2}^0 & \cdots & x_{i_k}^0 & x_{i_{k+1}}^0 \\ x_{i_1}^1 & x_{i_2}^1 & \cdots & x_{i_k}^1 & x_{i_{k+1}}^1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ x_{i_1}^{k-1} & x_{i_2}^{k-1} & \cdots & x_{i_k}^{k-1} & x_{i_{k+1}}^{k-1} \end{pmatrix},$$

then

$$\det A = \sum_{p=1}^{k+1} (-1)^{p+1} \det \begin{pmatrix} x_{i_1}^0 & x_{i_2}^0 & \cdots & \widehat{x_{i_p}^0} & \cdots & x_{i_k}^0 & x_{i_{k+1}}^0 \\ x_{i_1}^1 & x_{i_2}^1 & \cdots & \widehat{x_{i_p}^1} & \cdots & x_{i_k}^1 & x_{i_{k+1}}^1 \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\ x_{i_1}^{k-1} & x_{i_2}^{k-1} & \cdots & \widehat{x_{i_p}^{k-1}} & \cdots & x_{i_k}^{k-1} & x_{i_{k+1}}^{k-1} \end{pmatrix}$$

(by Laplace expansion along the first row of A). Comparing these two equalities, we find

$$\omega_k \left(\partial \left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k+1}} \right) \right) = \det A.$$

But the matrix A has two equal rows: In fact, its second row $\begin{pmatrix} x_{i_1}^0 & x_{i_2}^0 & \cdots & x_{i_k}^0 & x_{i_{k+1}}^0 \end{pmatrix}$ agrees with its first row $\begin{pmatrix} 1 & 1 & \cdots & 1 & 1 \end{pmatrix}$, since $x_j^0 = 1$ for all j. Hence, $\det A = 0$ (since a matrix with two equal rows must always have determinant 0). Thus,

$$\omega_k \left(\partial \left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k+1}} \right) \right) = \det A = 0.$$

As explained above, this proves Lemma 2.5. □

Next comes the most technical lemma in this note:

Lemma 2.6. Let $k > 0$. A monomial in $\mathcal{P}_n$ is called *nice* if it can be written as $x_{i_1}^1 x_{i_2}^2 \cdots x_{i_{k-1}}^{k-1}$ with $1 < i_1 < i_2 < \cdots < i_{k-1} \leq n$. Let $\eta : \mathcal{P}_n \to \Lambda^{k-1} N$ be the $\mathbf{k}$-linear map that

sends each nice monomial $x_{i_1}^1 x_{i_2}^2 \cdots x_{i_{k-1}}^{k-1}$ to $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}} \in \Lambda^{k-1} N$,

and sends each monomial that is not nice to 0.

(This is well-defined, since the monomials form a basis of the $\mathbf{k}$-module $\mathcal{P}_n$, and since a nice monomial can be written in the form $x_{i_1}^1 x_{i_2}^2 \cdots x_{i_{k-1}}^{k-1}$ uniquely.)

Then,

$$\varepsilon_1 \eta \omega_k = \mathrm{id} \ \text{ on } \varepsilon_1 \left(\Lambda^{k-1} N \right).$$

Proof. The **k**-module $\Lambda^{k-1}N$ is spanned by the vectors $e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}$ for all $(i_1, i_2, \ldots, i_{k-1}) \in [n]^{k-1}$ satisfying $i_1 < i_2 < \cdots < i_{k-1}$. Hence, its image $\varepsilon_1 \left(\Lambda^{k-1}N\right)$ is spanned by the images $\varepsilon_1 \left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}\right)$ of these vectors. Since $\varepsilon_1 \left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}\right) = e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}$ (by the definition of ε_1), we can restate this as follows: The image $\varepsilon_1 \left(\Lambda^{k-1}N\right)$ is spanned by the vectors $e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}$ for all $(i_1, i_2, \ldots, i_{k-1}) \in [n]^{k-1}$ satisfying $i_1 < i_2 < \cdots < i_{k-1}$.

However, many of these vectors vanish: Indeed, $e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}} = 0$ whenever $i_1 = 1$, since an exterior product with two equal factors is 0. Obviously, these vanishing vectors are unneccessary for spanning $\varepsilon_1 \left(\Lambda^{k-1}N\right)$. Hence, we can remove them from our list, and conclude that the image $\varepsilon_1 \left(\Lambda^{k-1}N\right)$ is spanned by the vectors $e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}$ for all $(i_1, i_2, \ldots, i_{k-1}) \in [n]^{k-1}$ satisfying $1 < i_1 < i_2 < \cdots < i_{k-1}$.

Thus, by linearity, it suffices (towards our goal of proving that $\varepsilon_1 \eta \omega_k = \mathrm{id}$ on $\varepsilon_1 \left(\Lambda^{k-1}N\right)$) to show that

$$(\varepsilon_1 \eta \omega_k) \left(e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}\right) = e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}$$

for all $(i_1, i_2, \ldots, i_{k-1}) \in [n]^{k-1}$ satisfying $1 < i_1 < i_2 < \cdots < i_{k-1}$. Let us do this.

Let $(i_1, i_2, \ldots, i_{k-1}) \in [n]^{k-1}$ be such that $1 < i_1 < i_2 < \cdots < i_{k-1}$. Then, the definition of ω_k yields

$$\omega_k \left(e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}\right) = V \left(x_1, x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}\right).$$

Setting $i_0 := 1$, we can rewrite this as

$$\begin{aligned} \omega_k \left(e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}\right) &= V \left(x_{i_0}, x_{i_1}, x_{i_2}, \ldots, x_{i_{k-1}}\right) \\ &= \det \left(x_{i_{u-1}}^{v-1}\right)_{u,v \in [k]} \\ &= \sum_{\sigma \in S_k} (-1)^{\sigma} \prod_{p=1}^{k} x_{i_{p-1}}^{\sigma(p)-1} \end{aligned} \tag{11}$$

by the Leibniz formula for the determinant.

The products $\prod\limits_{p=1}^{k} x_{i_{p-1}}^{\sigma(p)-1}$ that appear on the right hand side are monomials, but I claim that only one of them is nice: viz., the one obtained for $\sigma = \mathrm{id}$. Indeed, the latter product is

$$\prod_{p=1}^{k} x_{i_{p-1}}^{\mathrm{id}(p)-1} = \prod_{p=1}^{k} x_{i_{p-1}}^{p-1} = \underbrace{x_{i_0}^{0}}_{=1} x_{i_1}^{1} x_{i_2}^{2} \cdots x_{i_{k-1}}^{k-1} = x_{i_1}^{1} x_{i_2}^{2} \cdots x_{i_{k-1}}^{k-1},$$

which is clearly nice. Now let me show that none of the remaining products $\prod\limits_{p=1}^{k} x_{i_{p-1}}^{\sigma(p)-1}$ (with $\sigma \neq \mathrm{id}$) is nice. Indeed, a nice monomial cannot contain the variable x_1 at all, and must contain the exponents $1, 2, \ldots, k-1$ on variables with increasing subscripts (i.e., the larger the exponent, the larger the subscript). But a product $\prod_{p=1}^{k} x_{i_{p-1}}^{\sigma(p)-1}$ with $\sigma \neq \operatorname{id}$ cannot satisfy these two properties, since it either contains $x_{i_0} = x_1$ (when $\sigma(1) \neq 1$), or contains the other exponents $1, 2, \ldots, k-1$ in the "wrong order" (when $\sigma(1) = 1$, so that σ has an inversion among its values $\sigma(2), \sigma(3), \ldots, \sigma(k)$). Thus, a monomial of the form $\prod_{p=1}^{k} x_{i_{p-1}}^{\sigma(p)-1}$ with $\sigma \neq \operatorname{id}$ cannot be nice. Hence, such a monomial will always satisfy

$$\eta\left(\prod_{p=1}^{k} x_{i_{p-1}}^{\sigma(p)-1}\right) = 0 \qquad (12)$$

(since η kills all non-nice monomials, by definition).

Now, the map η is $\mathbf{k}$-linear. Hence,

$$\begin{aligned}
\eta\left(\sum_{\sigma \in S_k} (-1)^{\sigma} \prod_{p=1}^{k} x_{i_{p-1}}^{\sigma(p)-1}\right) &= \sum_{\sigma \in S_k} (-1)^{\sigma} \underbrace{\eta\left(\prod_{p=1}^{k} x_{i_{p-1}}^{\sigma(p)-1}\right)}_{\substack{=0 \text{ whenever } \sigma \neq \operatorname{id} \\ \text{(by (12))}}} \\
&= \underbrace{(-1)^{\operatorname{id}}}_{=1} \eta\left(\underbrace{\prod_{p=1}^{k} x_{i_{p-1}}^{\operatorname{id}(p)-1}}_{=x_{i_1}^1 x_{i_2}^2 \cdots x_{i_{k-1}}^{k-1}}\right) \\
&= \eta\left(x_{i_1}^1 x_{i_2}^2 \cdots x_{i_{k-1}}^{k-1}\right) = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}
\end{aligned}$$

(by the definition of η). In view of (11), this rewrites as

$$\eta\left(\omega_k\left(e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}\right)\right) = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}.$$

Hence,

$$\begin{aligned}
\varepsilon_1\left(\eta\left(\omega_k\left(e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}\right)\right)\right) &= \varepsilon_1\left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}\right) \\
&= e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}
\end{aligned}$$

(by the definition of ε_1). In other words,

$$\left(\varepsilon_1 \eta \omega_k\right)\left(e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}\right) = e_1 \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}}.$$

But this is precisely what remained to prove. So the proof of Lemma 2.6 is complete. □

Theorem 2.7. Let $k \in \mathbb{N}$. Then, the map $\omega_k : \Lambda^k N \to \mathcal{S}^{\lambda^k}$ is a surjective morphism of S_n-representations. Its kernel is

$$\operatorname{Ker}(\omega_k) = \operatorname{Ker}\left(\partial\mid_{\Lambda^k N}\right) = \partial\left(\Lambda^{k+1} N\right). \tag{13}$$

Thus, ω_k induces an isomorphism

$$\mathcal{S}^{\lambda^k} \cong (\operatorname{Coker} \partial)_k \qquad \text{of } S_n\text{-representations.} \tag{14}$$

Proof. We already know that ω_k is a morphism of S_n-representations.

From Proposition 2.1, we know that $\operatorname{Ker} \partial = \operatorname{Im} \partial$. Taking the k-th graded component of this equality, we obtain $\operatorname{Ker}\left(\partial\mid_{\Lambda^k N}\right) = \partial\left(\Lambda^{k+1} N\right)$.

Next, we shall prove that ω_k is surjective and that $\operatorname{Ker}(\omega_k) = \partial\left(\Lambda^{k+1} N\right)$.

We WLOG assume that $k > 0$, since the $k = 0$ case is easily done by hand (remember that $\omega_0 = 0$ and $\mathcal{S}^{\lambda^0} = 0$ by definition, and observe that $\partial\left(\Lambda^1 N\right)$ contains $\partial(e_1) = 1$ and thus is the whole $\Lambda^0 N$). Hence, $\mathcal{S}^{\lambda^k}$ is defined as the span of all $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$. Thus, the surjectivity of ω_k is obvious from its definition.

Proposition 2.1 yields $\partial\varepsilon_1 + \varepsilon_1\partial = \mathrm{id}$. Hence, $\Lambda^k N = \varepsilon_1\left(\Lambda^{k-1} N\right) + \partial\left(\Lambda^{k+1} N\right)$.

Lemma 2.5 says that $\omega_k\partial = 0$ on $\Lambda^{k+1} N$. In other words, $\partial\left(\Lambda^{k+1} N\right) \subseteq \operatorname{Ker}(\omega_k)$.

Lemma 2.6 shows that $\varepsilon_1\eta\omega_k = \mathrm{id}$ on $\varepsilon_1\left(\Lambda^{k-1} N\right)$ (where η is as defined in that lemma). Hence, the map ω_k is injective when restricted to $\varepsilon_1\left(\Lambda^{k-1} N\right)$ (since it has a left inverse $\varepsilon_1\eta$). In other words, $\operatorname{Ker}(\omega_k) \cap \varepsilon_1\left(\Lambda^{k-1} N\right) = 0$.

But a general fact (easy exercise) about modules says the following: If A, B, C are three $\mathbf{k}$-submodules of a $\mathbf{k}$-module M satisfying $M = A + B$ and $B \subseteq C$ and $C \cap A = 0$, then $B = C$. Applying this to $M = \Lambda^k N$ and $A = \varepsilon_1\left(\Lambda^{k-1} N\right)$ and $B = \partial\left(\Lambda^{k+1} N\right)$ and $C = \operatorname{Ker}(\omega_k)$, we obtain $\partial\left(\Lambda^{k+1} N\right) = \operatorname{Ker}(\omega_k)$ (since $\Lambda^k N = \varepsilon_1\left(\Lambda^{k-1} N\right) + \partial\left(\Lambda^{k+1} N\right)$ and $\partial\left(\Lambda^{k+1} N\right) \subseteq \operatorname{Ker}(\omega_k)$ and $\operatorname{Ker}(\omega_k) \cap \varepsilon_1\left(\Lambda^{k-1} N\right) = 0$). Hence, $\operatorname{Ker}(\omega_k) = \partial\left(\Lambda^{k+1} N\right)$ is proved. Combining this with $\operatorname{Ker}\left(\partial\mid_{\Lambda^k N}\right) = \partial\left(\Lambda^{k+1} N\right)$, we conclude that (13) holds.

By the homomorphism theorem, we have $\operatorname{Im}(\omega_k) \cong \Lambda^k N / \operatorname{Ker}(\omega_k)$ as S_n-representations (since ω_k is a morphism of S_n-representations). Since ω_k is surjective, we have $\operatorname{Im}(\omega_k) = \mathcal{S}^{\lambda^k}$ and thus

$$\mathcal{S}^{\lambda^k} = \operatorname{Im}(\omega_k) \cong \Lambda^k N / \underbrace{\operatorname{Ker}(\omega_k)}_{=\partial\left(\Lambda^{k+1} N\right)} = \Lambda^k N / \partial\left(\Lambda^{k+1} N\right) = (\operatorname{Coker} \partial)_k$$

as S_n-representations. Thus we have proved (14), and Theorem 2.7 is proven. □

We can now show that Peel's maps f_k are well-defined and form an exact sequence, and in fact form a commutative diagrams with the maps $(-1)^k \varepsilon$ on ΛN and the morphisms ω_k:

Theorem 2.8. Assume that $n = 0$ in $\mathbf{k}$. Then, for each $k > 0$, the map

$$\begin{aligned} f_k : \mathcal{S}^{\lambda^k} &\to \mathcal{S}^{\lambda^{k+1}}, \\ V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right) &\mapsto \sum_{s \in [n]} V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_s\right) \end{aligned}$$

is a well-defined morphism of S_n-representations. Let furthermore $f_0 : \mathcal{S}^{\lambda^0} \to \mathcal{S}^{\lambda^1}$ be the zero map 0. Then, the diagram

$$\begin{array}{ccc} \Lambda^k N & \xrightarrow{(-1)^k \varepsilon} & \Lambda^{k+1} N \\ {\scriptstyle \omega_k}\downarrow & & \downarrow{\scriptstyle \omega_{k+1}} \\ \mathcal{S}^{\lambda^k} & \xrightarrow[f_k]{} & \mathcal{S}^{\lambda^{k+1}} \end{array} \tag{15}$$

commutes for each $k \in \mathbb{N}$. Thus, the sequence

$$\cdots \xrightarrow{f_{k-2}} \mathcal{S}^{\lambda^{k-1}} \xrightarrow{f_{k-1}} \mathcal{S}^{\lambda^k} \xrightarrow{f_k} \mathcal{S}^{\lambda^{k+1}} \xrightarrow{f_{k+1}} \cdots \tag{16}$$

is exact and has a $\mathbf{k}$-linear (but not S_n-equivariant) chain contraction.

Proof. First, we note that the S_n-action is not relevant to any claims of the theorem, except for the easy claim that f_k is a morphism of S_n-representations. Thus, we can forget about this action now (although this does not simplify much, just taking some minor cargo off our backs).

Theorem 2.7 shows that for each $k \in \mathbb{N}$, the map $\omega_k : \Lambda^k N \to \mathcal{S}^{\lambda^k}$ is surjective and has kernel $\operatorname{Ker}(\omega_k) = \partial\left(\Lambda^{k+1} N\right)$. Thus, this map gives rise to a canonical isomorphism

$$\begin{aligned} \omega'_k : \Lambda^k N \diagup \partial\left(\Lambda^{k+1} N\right) &\overset{\cong}{\to} \mathcal{S}^{\lambda^k}, \\ \overline{v} &\mapsto \omega_k(v) \end{aligned}$$

(where $\overline{v}$ denotes the projection of a vector v onto the quotient). Since we have $\Lambda^k N \diagup \partial\left(\Lambda^{k+1} N\right) = (\operatorname{Coker} \partial)_k$, we can rewrite this as

$$\begin{aligned} \omega'_k : (\operatorname{Coker} \partial)_k &\overset{\cong}{\to} \mathcal{S}^{\lambda^k}, \\ \overline{v} &\mapsto \omega_k(v). \end{aligned}$$

Using these isomorphisms ω'_k, we can turn the morphisms $\varepsilon' : (\operatorname{Coker} \partial)_k \to (\operatorname{Coker} \partial)_{k+1}$ from Proposition 2.3 into morphisms $\varepsilon''_k : \mathcal{S}^{\lambda^k} \to \mathcal{S}^{\lambda^{k+1}}$ so that the diagram

$$\begin{array}{ccc} (\operatorname{Coker}\partial)_k & \xrightarrow{\varepsilon'} & (\operatorname{Coker}\partial)_{k+1} \\ \omega'_k \downarrow \cong & & \cong \downarrow \omega'_{k+1} \\ \mathcal{S}^{\lambda^k} & \xrightarrow[\varepsilon''_k]{} & \mathcal{S}^{\lambda^{k+1}} \end{array} \tag{17}$$

commutes (explicitly, we set $\varepsilon''_k := \omega'_{k+1}\varepsilon'\left(\omega'_k\right)^{-1}$ on $\mathcal{S}^{\lambda^k}$). Consider these morphisms ε''_k. Let us furthermore define a new morphism

$$g_k := (-1)^k \varepsilon''_k : \mathcal{S}^{\lambda^k} \to \mathcal{S}^{\lambda^{k+1}} \qquad \text{for each } k \in \mathbb{N}.$$

Thus, g_k differs from ε''_k only in the sign factor $(-1)^k$. Hence, the commutative diagram (17) yields a commutative diagram

$$\begin{array}{ccc} (\operatorname{Coker}\partial)_k & \xrightarrow{(-1)^k\varepsilon'} & (\operatorname{Coker}\partial)_{k+1} \\ \omega'_k \downarrow \cong & & \cong \downarrow \omega'_{k+1} \\ \mathcal{S}^{\lambda^k} & \xrightarrow[g_k]{} & \mathcal{S}^{\lambda^{k+1}} \end{array} \tag{18}$$

(obtained from it by scaling both horizontal arrows by $(-1)^k$).

We claim that the new maps g_k are precisely the maps f_k defined in the theorem (and, in particular, the latter maps f_k are well-defined). Indeed, this is obvious for $k = 0$, so let us take $k > 0$. Let $(i_1, i_2, \ldots, i_k) \in [n]^k$. Then,

$$\begin{aligned} & V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right) \\ & = \omega_k\left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\right) && \text{(by the definition of } \omega_k\text{)} \\ & = \omega'_k\left(\overline{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}}\right) && \left(\text{by the definition of } \omega'_k\right); \end{aligned}$$

thus,

$$\begin{aligned} & g_k\left(V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)\right) \\ & = g_k\left(\omega'_k\left(\overline{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}}\right)\right) \\ & = \omega'_{k+1}\left((-1)^k \varepsilon'\left(\overline{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}}\right)\right) \end{aligned}$$

(since the diagram (18) commutes). Since

$$\begin{aligned}
&(-1)^k \varepsilon' \left(\overline{e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}}\right) \\
&= (-1)^k \overline{\varepsilon\left(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}\right)} \qquad \left(\text{by the definition of } \varepsilon'\right) \\
&= (-1)^k \overline{e \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}} \qquad (\text{by the definition of } \varepsilon) \\
&= (-1)^k \overline{\left(\sum_{s\in[n]} e_s\right) \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}} \qquad \left(\text{since } e = \sum_{s\in[n]} e_s\right) \\
&= \overline{\sum_{s\in[n]} \underbrace{(-1)^k e_s \wedge e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k}}_{=e_{i_1}\wedge e_{i_2}\wedge\cdots\wedge e_{i_k}\wedge e_s}} \\
&= \overline{\sum_{s\in[n]} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \wedge e_s},
\end{aligned}$$

this rewrites as

$$\begin{aligned}
&g_k\left(V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)\right) \\
&= \omega'_{k+1}\left(\overline{\sum_{s\in[n]} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \wedge e_s}\right) \\
&= \omega_{k+1}\left(\sum_{s\in[n]} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_k} \wedge e_s\right) \qquad \left(\text{by the definition of } \omega'_{k+1}\right) \\
&= \sum_{s\in[n]} V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_s\right) \qquad \left(\text{by the definition of } \omega_{k+1}\right).
\end{aligned}$$

So we have shown that $g_k : \mathcal{S}^{\lambda^k} \to \mathcal{S}^{\lambda^{k+1}}$ is a $\mathbf{k}$-linear map that sends each $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ to $\sum_{s\in[n]} V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}, x_s\right)$. But this is exactly what the map f_k is supposed to do. Thus, it follows that the map f_k is well-defined (it is unique since the $V\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}\right)$ span $\mathcal{S}^{\lambda^k}$), and that $g_k = f_k$.

We have thus proved that for each $k \in \mathbb{N}$, the map f_k is well-defined and satisfies $g_k = f_k$. As we said, it is easy to show that f_k is a morphism of S_n-representations.

Now, the commutative diagram (18) rewrites as

$$\begin{array}{ccc}
(\operatorname{Coker} \partial)_k & \xrightarrow{(-1)^k \varepsilon'} & (\operatorname{Coker} \partial)_{k+1} \\
\omega'_k \big\downarrow \cong & & \cong \big\downarrow \omega'_{k+1} \\
\mathcal{S}^{\lambda^k} & \xrightarrow[f_k]{} & \mathcal{S}^{\lambda^{k+1}}
\end{array} \qquad (19)$$

(since $g_k = f_k$). Now, we consider the diagram

$$\begin{array}{ccc} \Lambda^k N & \xrightarrow{(-1)^k \varepsilon} & \Lambda^{k+1} N \\ \downarrow & & \downarrow \\ (\operatorname{Coker} \partial)_k & \xrightarrow{(-1)^k \varepsilon'} & (\operatorname{Coker} \partial)_{k+1} \\ \omega'_k \downarrow \cong & & \cong \downarrow \omega'_{k+1} \\ \mathcal{S}^{\lambda^k} & \xrightarrow[f_k]{} & \mathcal{S}^{\lambda^{k+1}} \end{array} \qquad (20)$$

(where the two topmost vertical arrows are canonical projections $\Lambda N \to \operatorname{Coker} \partial$). This diagram is commutative, because the top square commutes (indeed, it is just the diagram (10) with both horizontal arrows scaled by $(-1)^k$) and the bottom square commutes (this square is just the diagram (19)). Composing the vertical arrows in this diagram (and recalling that $\omega'_k (\overline{v}) = \omega_k (v)$ for each $v \in \Lambda^k N$), we obtain the commutative diagram

$$\begin{array}{ccc} \Lambda^k N & \xrightarrow{(-1)^k \varepsilon} & \Lambda^{k+1} N \\ \omega_k \downarrow & & \downarrow \omega_{k+1} \\ \mathcal{S}^{\lambda^k} & \xrightarrow[f_k]{} & \mathcal{S}^{\lambda^{k+1}} \end{array} .$$

Thus we have shown that the diagram (15) commutes.

Recall from Theorem 2.4 that the sequence

$$\cdots \xrightarrow{\varepsilon'} (\operatorname{Coker} \partial)_{k-1} \xrightarrow{\varepsilon'} (\operatorname{Coker} \partial)_k \xrightarrow{\varepsilon'} (\operatorname{Coker} \partial)_{k+1} \xrightarrow{\varepsilon'} \cdots$$

is exact and has a **k**-linear chain contraction. Hence, the same is true of the sequence

$$\cdots \xrightarrow{(-1)^{k-2}\varepsilon'} (\operatorname{Coker} \partial)_{k-1} \xrightarrow{(-1)^{k-1}\varepsilon'} (\operatorname{Coker} \partial)_k \xrightarrow{(-1)^{k}\varepsilon'} (\operatorname{Coker} \partial)_{k+1} \xrightarrow{(-1)^{k+1}\varepsilon'} \cdots$$

(because the exactness of a sequence is preserved when we scale every other arrow by -1, and the same holds for the existence of a chain contraction; indeed, this is just the shift functor $A \mapsto A[1]$ on chain complexes). Thus, the same is true of the sequence

$$\cdots \xrightarrow{f_{k-2}} \mathcal{S}^{\lambda^{k-1}} \xrightarrow{f_{k-1}} \mathcal{S}^{\lambda^k} \xrightarrow{f_k} \mathcal{S}^{\lambda^{k+1}} \xrightarrow{f_{k+1}} \cdots ,$$

because the commutative diagram (19) reveals that these two sequences are isomorphic (with isomorphism given by $\omega'_k : (\operatorname{Coker} \partial)_k \xrightarrow{\cong} \mathcal{S}^{\lambda^k}$). This finishes the proof of Theorem 2.8. □

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