The hook length formula [talk slides]

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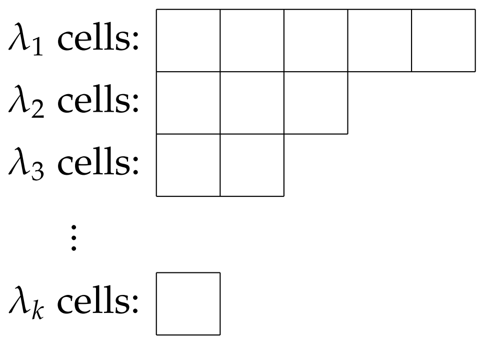
We discuss the hook length formula and some related results.

1. The hook length formula

1.1. Recalling definitions

- We follow the notations in the notes.
- Each partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ has a Young diagram $Y(\lambda)$ associated to it:



If two partitions λ and μ satisfy $Y(\mu) \subseteq Y(\lambda)$ (or, short: $\mu \subseteq \lambda$), then the skew Young diagram $Y(\lambda/\mu)$ is defined to be $Y(\lambda) \setminus Y(\mu)$.

- Let D be a diagram. A *standard tableau* of shape D is a bijective filling of D with the numbers $1, 2, \ldots, n$ such that the entries increase left-to-right along rows and top-to-bottom along columns.

Examples:

straight-shaped

1	3	7
2	4	
5	6	

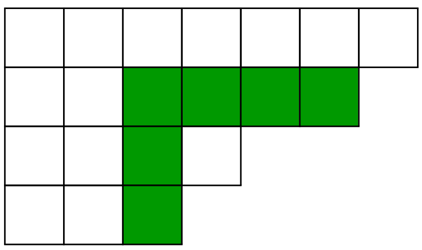
skew-shaped

		3	5	6
	2	4		
1	7			

- We will only consider straight shapes $Y(\lambda)$ and skew shapes $Y(\lambda/\mu)$ in this talk.
- We say "tableau of shape λ" instead of "tableau of shape $Y(\lambda)$". Likewise for λ/μ.
- If c is a cell of $Y(\lambda)$, then we define the *hook* $H_\lambda(c)$ of this cell in λ to be

$$H_\lambda(c) := \{c\} \cup \{\text{all cells of } Y(\lambda) \text{ due east of } c\} \cup \{\text{all cells of } Y(\lambda) \text{ due south of } c\}.$$

For example, if $\lambda = (7,6,4,3)$ and $c = (2,3)$, then $H_\lambda(c)$ is the set of all green cells here:

 .

1.2. The hook length formula

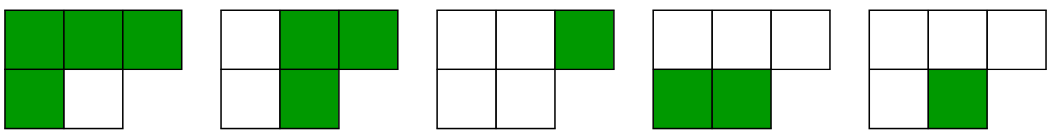
- **The hook length formula:** For any partition λ of n, we have

$$(\#\text{ of standard tableaux of shape } \lambda) = \frac{n!}{\prod_{c \in Y(\lambda)} |H_\lambda(c)|}.$$

- **Example:** If $\lambda = (3,2)$, then this becomes

$$(\#\text{ of standard tableaux of shape } (3,2)) = \frac{5!}{4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 5.$$

Here are the hooks of all five cells:



and here are the five standard tableaux:

1	2	3
4	5	

1	2	4
3	5	

1	2	5
3	4	

1	3	4
2	5	

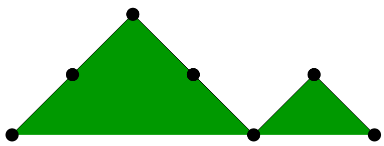
1	3	5
2	4	

.

- **Example:** Let $\lambda = (m, m)$ (so that $n = 2m$). Then, the hook length formula becomes

$$\begin{aligned} &(\text{\# of standard tableaux of shape } (m, m)) \\ &= \frac{(2m)!}{((m+1)\,m\,(m-1)\cdots 2)\cdot(m\,(m-1)\,(m-2)\cdots 1)} \\ &= \frac{1}{m+1}\binom{2m}{m}. \end{aligned}$$

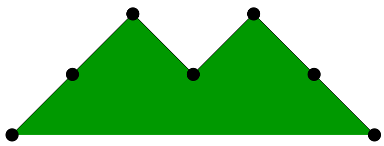
Looks familiar? This is the m-th Catalan number C_m. And indeed, there is a bijection to Dyck paths:



$\longleftrightarrow$

1	2	5
3	4	6

.



$\longleftrightarrow$

1	2	4
3	5	6

.

1.3. How not to prove the hook length formula

- **The hook length formula:** For any partition λ of n, we have

$$(\text{\# of standard tableaux of shape } \lambda) = \frac{n!}{\prod_{c \in Y(\lambda)} |H_\lambda(c)|}.$$

- Isn't this easy?
  - There are $n!$ bijective fillings of $Y(\lambda)$ with the numbers $1, 2, \ldots, n$.

  – Such a filling is a standard tableau if and only if for each cell $c \in Y(\lambda)$, the entry in cell c is smaller than all other entries in its hook $H_\lambda(c)$.

  e.g.:

		a	b_1	b_2	b_3	
		b_4				
		b_5				

$a < b_1, b_2, b_3, b_4, b_5.$

  The probability for this is $\dfrac{1}{|H_\lambda(c)|}$.

  – Multiplying these probabilities over all cells c, we get $\dfrac{1}{\prod\limits_{c \in Y(\lambda)} |H_\lambda(c)|}$, thus the formula.

- Alas, it is not this easy: The events are not independent, so we cannot just multiply their probabilities. That we got the right result is a surprise and needs proof!

1.4. Some context

- **Standard basis theorem** (Theorem 5.9.1 in the notes)**:** The Specht module $\mathcal{S}^\lambda := \mathcal{S}^{Y(\lambda)}$ has a basis indexed by the standard tableaux of shape λ.
- Thus, the hook length formula gives the rank (= dimension) of this Specht module.
- Actually, the standard basis theorem holds for skew Young diagrams as well, but the hook length formula does not apply to them.
- This all is unnecessary to understand and prove the hook length formula. But it is the very context in which tableaux were originally defined by Alfred Young ca. 1902.

1.5. Some history

- Young never stated the hook length formula in its present form. What he found was the following (1928):

- **Young's quotient formula:** Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ be any partition with $|\lambda| = n$. Let $\ell_i := \lambda_i + k - i$ for each $i \in [k]$. Then, the # of standard tableaux of shape $Y(\lambda)$ is

$$\frac{n!}{\ell_1! \cdot \ell_2! \cdot \cdots \cdot \ell_k!} \prod_{1 \leq i < j \leq k} (\ell_i - \ell_j).$$

- He gave essentially two proofs.

- One proof (1901–1928) is quite intricate, using Specht modules and long computations with Young symmetrizers. The main idea is to expand the squared Young symmetrizer $\mathbf{E}_T^2$ in two ways: once using the Young symmetrizer theorem $\mathbf{E}_T^2 = \frac{n!}{f^\lambda}\mathbf{E}_T$ (Theorem 5.11.3 in the notes), and again by writing

$$\mathbf{E}_T^2 = \nabla^-_{\operatorname{Col} T} \nabla_{\operatorname{Row} T} \nabla^-_{\operatorname{Col} T} \nabla_{\operatorname{Row} T}$$

  and gradually "dissolving" the inner $\nabla^-_{\operatorname{Col} T}$ factor by factoring it and cancelling pieces of it against the adjacent $\nabla_{\operatorname{Row} T}$ factors.

  Note that Young knew the standard basis theorem and even some of the Garnir relations, though his proof of the former was much more complicated.

  A variant of this proof appears in §5.21 (last section of Chapter 5, in case the numbering shifts) of the notes.

- The other proof (1928) is pretty elementary, and I will sketch it.

- The modern form of the hook length formula was found in 1953:

> One Thursday in May of 1953, [Gilbert de Beauregard] Robinson was visiting [James Sutherland] Frame at Michigan State University. Discussing the work of Staal (a student of Robinson), Frame was led to conjecture the hook formula. At first Robinson could not believe that such a simple formula existed, but after trying some examples he became convinced, and together they proved the identity. On Saturday they went to the University of Michigan, where Frame presented their new result after a lecture by Robinson. This surprised [Robert McDowell] Thrall, who was in

the audience, because he had just proved the same result on the same day!

(Bruce Sagan)

- This is perhaps less surprising once you realize that Young's quotient formula was known by then, and the leap to the hook length formula was not that large: you just need to show that

$$\frac{1}{\prod\limits_{c \in Y(\lambda)} |H_\lambda(c)|} = \frac{1}{\ell_1! \cdot \ell_2! \cdot \cdots \cdot \ell_k!} \prod_{1 \leq i < j \leq k} (\ell_i - \ell_j),$$

  which can be done by induction on k (removing the first row of $Y(\lambda)$).

1.6. Proofs galore

- The hook length formula is the quadratic reciprocity of algebraic combinatorics: every text gives a proof. Quite a few of them are different: e.g.
  - Young 1901–1928 using Young symmetrizers;
  - Young 1928 using rational functions (sketched below);
  - James 1978 using determinants;
  - Greene, Nijenhuis, Wilf 1979 using discrete probability;
  - Novelli, Pak, Stojanovskii 1997 by an intricate (multi)bijection;
  - ...

  See §11.2 in Igor Pak's arXiv:2209.06142v1 for a partial taxonomy.

1.7. Young's rational functions proof

- Let me sketch Young's second proof, from his 1928 paper *On Quantitative Substitutional Analysis (Third Paper)*.

  We shall prove Young's quotient formula; as we mentioned, the hook length formula follows easily.

- We let

$$f^{\lambda} := (\text{\# of standard tableaux of shape } \lambda) \qquad \text{and}$$
$$g^{\lambda} := \frac{n!}{\ell_1! \cdot \ell_2! \cdot \cdots \cdot \ell_k!} \prod_{1 \leq i < j \leq k} (\ell_i - \ell_j),$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ is a partition with $|\lambda| = n$, and where $\ell_i := \lambda_i + k - i$ for each $i \in [k]$.

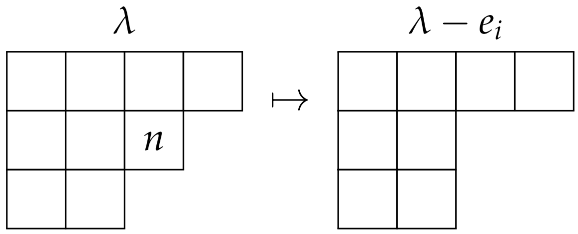
Our goal is to show that $f^{\lambda} = g^{\lambda}$.

- Assume at first that all rows of $Y(\lambda)$ have distinct lengths, i.e., that

$$\lambda_1 > \lambda_2 > \cdots > \lambda_k.$$

A standard tableau of shape λ must have its entry n in some cell that is at the end of one of its rows, say the i-th row. Removing that cell leaves us with a standard tableau of shape

$$\lambda - e_i := (\lambda_1, \lambda_2, \ldots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_k).$$

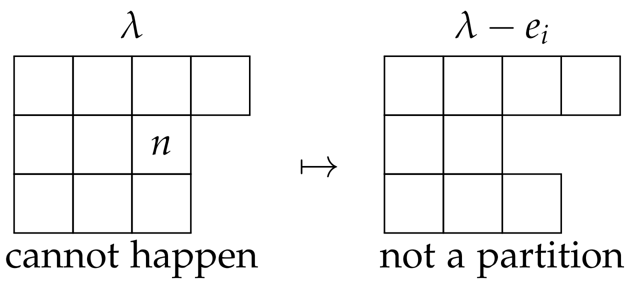


This is a bijection. Thus,

$$f^{\lambda} = f^{\lambda - e_1} + f^{\lambda - e_2} + \cdots + f^{\lambda - e_k},$$

which is a recursion for f^{λ}.

- This is also true without the "distinct lengths" assumption, as long as we understand $f^{\lambda - e_i}$ to be 0 when $\lambda - e_i$ is not a partition (i.e., when $\lambda_i = \lambda_{i+1}$). This makes perfect sense: The entry n cannot lie at the end of the i-th row in this case.



- If we can prove the same recursion for g^λ, that is,

$$g^\lambda \stackrel{?}{=} g^{\lambda - e_1} + g^{\lambda - e_2} + \cdots + g^{\lambda - e_k},$$

then we will be done by induction. So this recursion is our new goal. Note that we keep trailing zeroes in our partition (e.g., we don't simplify $(3,2,0)$ to $(3,2)$), so that k remains fixed.

- Expanding both sides of

$$g^\lambda \stackrel{?}{=} g^{\lambda - e_1} + g^{\lambda - e_2} + \cdots + g^{\lambda - e_k}$$

using

$$g^\lambda = \frac{n!}{\ell_1! \cdot \ell_2! \cdot \cdots \cdot \ell_k!} \prod_{1 \leq i < j \leq k} \left(\ell_i - \ell_j\right) \qquad \text{(by definition)};$$

$$g^{\lambda - e_p} = \frac{(n-1)!}{\ell_1! \cdot \ell_2! \cdot \cdots \cdot \ell_{p-1}! \cdot \left(\ell_p - 1\right)! \cdot \ell_{p+1}! \cdot \cdots \cdot \ell_k!} \prod_{1 \leq i < j \leq k} \begin{cases} \ell_i - \ell_j, & \text{if } i, j \neq p; \\ \left(\ell_p - 1\right) - \ell_j, & \text{if } i = p; \\ \ell_i - \left(\ell_p - 1\right), & \text{if } j = p, \end{cases} \qquad \text{(likewise)},$$

we see that most factors cancel, and we are left with proving that

$$n \stackrel{?}{=} \sum_{p=1}^{k} \ell_p \left(\prod_{j=p+1}^{k} \frac{\left(\ell_p - 1\right) - \ell_j}{\ell_p - \ell_j} \right) \left(\prod_{i=1}^{p-1} \frac{\ell_i - \left(\ell_p - 1\right)}{\ell_i - \ell_p} \right).$$

In view of $n = \sum\limits_{p=1}^{k} \lambda_p = \sum\limits_{p=1}^{k} \ell_p - \dbinom{k}{2}$, this rewrites as

$$\sum_{p=1}^{k} \ell_p - \binom{k}{2} \stackrel{?}{=} \sum_{p=1}^{k} \ell_p \left(\prod_{j=p+1}^{k} \frac{\left(\ell_p - 1\right) - \ell_j}{\ell_p - \ell_j} \right) \left(\prod_{i=1}^{p-1} \frac{\ell_i - \left(\ell_p - 1\right)}{\ell_i - \ell_p} \right).$$

- This is an identity between rational functions in $\ell_1, \ell_2, \ldots, \ell_k$. Thus, if it is to be true for all positive integers $\ell_1 > \ell_2 > \cdots > \ell_k$, then it must be true for all $\ell_1, \ell_2, \ldots, \ell_k \in \mathbb{C}$. This means that we can forget what our $\ell_1, \ell_2, \ldots, \ell_k$ are and just focus on this identity as an algebraic identity!

- There are several ways to prove this identity. Here is Young's:

- First combine the two products on the RHS:

$$\left(\prod_{j=p+1}^{k} \frac{(\ell_p - 1) - \ell_j}{\ell_p - \ell_j}\right) \left(\prod_{i=1}^{p-1} \frac{\ell_i - (\ell_p - 1)}{\ell_i - \ell_p}\right) = \prod_{\substack{i \in [k]; \\ i \neq p}} \frac{(\ell_p - 1) - \ell_i}{\ell_p - \ell_i}.$$

  So we need to show that

$$\sum_{p=1}^{k} \ell_p - \binom{k}{2} \stackrel{?}{=} \sum_{p=1}^{k} \ell_p \prod_{\substack{i \in [k]; \\ i \neq p}} \frac{(\ell_p - 1) - \ell_i}{\ell_p - \ell_i}.$$

- Using the polynomial

$$f(x) := (x - \ell_1)(x - \ell_2) \cdots (x - \ell_k),$$

  we can write

$$\prod_{\substack{i \in [k]; \\ i \neq p}} \frac{(\ell_p - 1) - \ell_i}{\ell_p - \ell_i} = -\frac{f(\ell_p - 1)}{\prod_{\substack{i \in [k]; \\ i \neq p}} (\ell_p - \ell_i)}.$$

  So we need to show that

$$\sum_{p=1}^{k} \ell_p - \binom{k}{2} \stackrel{?}{=} -\sum_{p=1}^{k} \ell_p \frac{f(\ell_p - 1)}{\prod_{\substack{i \in [k]; \\ i \neq p}} (\ell_p - \ell_i)}.$$

- **Sylvester's identity** says that
  - every polynomial $\Phi \in \mathbb{C}[x]$ of degree $< k - 1$ satisfies

$$\sum_{p=1}^{k} \frac{\Phi(\ell_p)}{\prod_{\substack{i \in [k]; \\ i \neq p}} (\ell_p - \ell_i)} = 0.$$

  - slightly more generally: if $\Phi \in \mathbb{C}[x]$ is a polynomial of degree $\leq k - 1$, and if φ_{k-1} is its x^{k-1}-coefficient, then

$$\sum_{p=1}^{k} \frac{\Phi(\ell_p)}{\prod_{\substack{i \in [k]; \\ i \neq p}} (\ell_p - \ell_i)} = \varphi_{k-1}.$$

[*Proof:* Lagrange interpolation says that

$$\sum_{p=1}^{k} \Phi\left(\ell_p\right) \frac{\prod\limits_{\substack{i\in[k];\\ i\neq p}} \left(x-\ell_i\right)}{\prod\limits_{\substack{i\in[k];\\ i\neq p}} \left(\ell_p-\ell_i\right)} = \Phi\left(x\right).$$

Compare coefficients before x^{k-1}.]

- We cannot apply this to the polynomial

$$\begin{aligned} \Phi &= xf\left(x-1\right) \\ &= x\left(x-1-\ell_1\right)\left(x-1-\ell_2\right)\cdots\left(x-1-\ell_k\right) \end{aligned}$$

directly, since its degree is too large ($k+1$ rather than $\leq k-1$). But we can instead apply it to the "second-order approximation"

$$\Phi := x\left(f\left(x-1\right)-f\left(x\right)+f'\left(x\right)\right),$$

which is a polynomial of degree $\leq k-1$ because subtracting $f\left(x\right)-f'\left(x\right)$ has cancelled the two highest terms (cf. Taylor series: $f\left(x-1\right)=f\left(x\right)-f'\left(x\right)+\frac{1}{2}f''\left(x\right)-\frac{1}{6}f'''\left(x\right)\pm\cdots$). Thus we get

$$\sum_{p=1}^{k} \frac{\ell_p\left(f\left(\ell_p-1\right)-f\left(\ell_p\right)+f'\left(\ell_p\right)\right)}{\prod\limits_{\substack{i\in[k];\\ i\neq p}} \left(\ell_p-\ell_i\right)} = \varphi_{k-1} = \frac{k\left(k-1\right)}{2},$$

since

$$f\left(x-1\right)-f\left(x\right)+f'\left(x\right)=\frac{1}{2}f''\left(x\right)-\frac{1}{6}f'''\left(x\right)\pm\cdots.$$

Subtracting

$$\sum_{p=1}^{k} \frac{-\ell_p f\left(\ell_p\right)}{\prod\limits_{\substack{i\in[k];\\ i\neq p}} \left(\ell_p-\ell_i\right)} = 0 \qquad \left(\text{since } f\left(\ell_p\right)=0\right) \qquad \text{and}$$

$$\sum_{p=1}^{k} \frac{\ell_p f'\left(\ell_p\right)}{\prod\limits_{\substack{i\in[k];\\ i\neq p}} \left(\ell_p-\ell_i\right)} = \sum_{p=1}^{k} \ell_p \qquad \left(\text{since } f'\left(\ell_p\right)=\prod\limits_{\substack{i\in[k];\\ i\neq p}} \left(\ell_p-\ell_i\right)\right),$$

we obtain

$$\sum_{p=1}^{k} \ell_p \frac{f(\ell_p - 1)}{\prod\limits_{\substack{i\in[k];\\ i\neq p}} (\ell_p - \ell_i)} = \frac{k(k-1)}{2} - 0 - \sum_{p=1}^{k} \ell_p$$
$$= \binom{k}{2} - \sum_{p=1}^{k} \ell_p$$

and thus

$$\sum_{p=1}^{k} \ell_p - \binom{k}{2} = -\sum_{p=1}^{k} \ell_p \frac{f(\ell_p - 1)}{\prod\limits_{\substack{i\in[k];\\ i\neq p}} (\ell_p - \ell_i)},$$

as desired.

- We are almost done. We need to show that $f^{\lambda - e_p} = g^{\lambda - e_p}$ holds when $\lambda - e_p$ is not a partition. (This is essentially the base case of our recursion, along with the easy case $\lambda = (0, 0, \ldots, 0)$.)

  This is easy: $f^{\lambda - e_p} = 0$ by definition, whereas

$$g^{\lambda - e_p} = \frac{(n-1)!}{\ell_1! \cdot \ell_2! \cdot \cdots \cdot \ell_{p-1}! \cdot (\ell_p - 1)! \cdot \ell_{p+1}! \cdot \cdots \cdot \ell_k!} \prod_{1\leq i<j\leq k} \begin{cases} \ell_i - \ell_j, & \text{if } i, j \neq p; \\ (\ell_p - 1) - \ell_j, & \text{if } i = p; \\ \ell_i - (\ell_p - 1), & \text{if } j = p, \end{cases}$$
$$= 0$$

  since the factor for $(i, j) = (p, p+1)$ is 0 (check it!).

- Thus the proof of the hook length formula is complete.

- This proof still keeps getting rediscovered: e.g., Glass and Ng, *A Simple Proof of the Hook Length Formula*, The American Mathematical Monthly **111** (2004), pp. 700–704. (With minor variations: e.g., they use the residue theorem instead of Lagrange interpolation.)

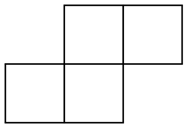
2. Naruse's skew hook length formula

2.1. The Young–Frobenius–Aitken determinant formula

- Now what about skew shapes?

$$(\text{\# of standard tableaux of shape } \lambda/\mu) = ???$$

- **Example:** Let $\lambda/\mu = (3,2)\ /\ (1)$, so that

$Y(\lambda/\mu) =$  .

Then,

$$(\text{\# of standard tableaux of shape } \lambda/\mu) = 5.$$

This is not $\frac{4!}{\text{an integer}}$, so don't expect any $\frac{n!}{\prod}$-type formula any more.

- For a long time, the best formula known was the **Young–Frobenius–Aitken determinantal formula**

$$(\text{\# of standard tableaux of shape } \lambda/\mu) = n! \cdot \det \left(\frac{1}{(\lambda_i - \mu_j - i + j)!} \right)_{i,j \in [k]},$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$ and where we set $\frac{1}{m!} := 0$ for $m < 0$.

This goes back to Frobenius (1900) as a formula for $\dim \left(\mathcal{S}^{\lambda/\mu} \right)$ (at least for $\mu = \varnothing$), then made combinatorial by Young (1935?) as he realized that $\mathcal{S}^{\lambda/\mu}$ has a standard tableau basis.

For a modern proof, see e.g. Theorem 5.6 in Adin/Roichman, *Enumeration of Standard Young Tableaux*, arXiv:1408.4497v2 (also a lot more there).

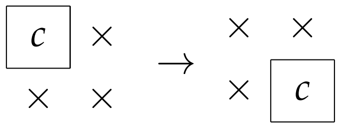
Alternatively, if you know the Jacobi–Trudi formula for the Schur function $s_{\lambda/\mu}$, you can evaluate it at $\underbrace{1, 1, \ldots, 1}_{k \text{ ones}}, 0, 0, \ldots, 0$ and consider the leading term as $k \to \infty$. (The larger k gets, the less likely is a semistandard tableau with entries in $\{1, 2, \ldots, k\}$ to have two equal entries!)

- Recently, two formulas generalizing the hook length formula came into existence. I will introduce the **Naruse hook length formula** (Naruse 2014).

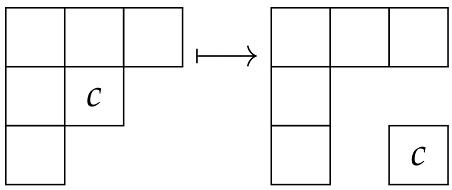
2.2. Excited diagrams

- Let D be any diagram.
- An *excited move* for a cell $c = (i, j) \in D$ means moving this cell from (i, j) to $(i + 1, j + 1)$.

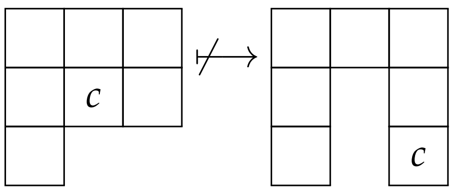
  This is allowed **only** if the three cells marked $\times$ in the picture below (that is, $(i + 1, j)$, $(i, j + 1)$, $(i + 1, j + 1)$) are not in D.



- **Examples:**



allowed.



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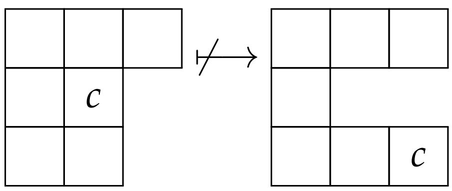

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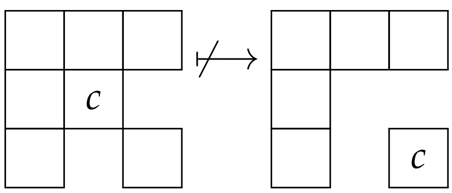

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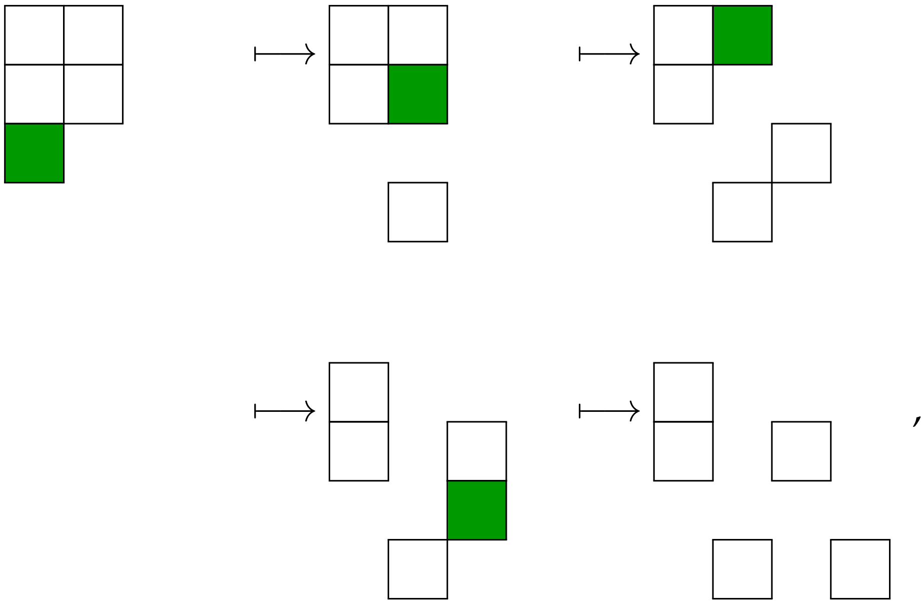
- An *excitation* of a diagram D is a diagram obtained from D by a sequence of excited moves. (This sequence can be empty, so D itself is an excitation of D.)

- **Example:** Start with $Y(2,2,1)$ and make some excited moves:



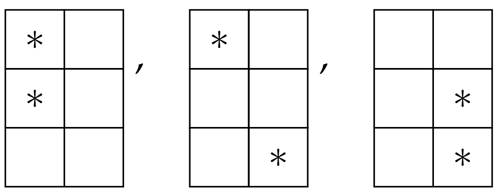
so each of these five diagrams is an excitation of the first.

2.3. Naruse's skew hook length formula

- **Naruse's skew hook length formula:** Let λ and μ be two partitions with $\mu \subseteq \lambda$ and $|Y(\lambda/\mu)| = n$. Then,

$$\begin{aligned}&(\text{\# of standard tableaux of shape } \lambda/\mu) \\ &= n! \sum_{\substack{E \text{ is an excitation of } Y(\mu); \\ E \subseteq Y(\lambda)}} \prod_{c \in Y(\lambda) \setminus E} \frac{1}{|H_\lambda(c)|}.\end{aligned}$$

- **Example:** If $\lambda = (2,2,2)$ and $\mu = (1,1)$, then the E's in the sum are



(where the cells in E are now marked with $*$s). Thus,

$$\begin{aligned}
&(\text{\# of standard tableaux of shape } \lambda/\mu) \\
&= 4! \cdot \left(\frac{1}{3\cdot 2\cdot 1\cdot 2} + \frac{1}{3\cdot 2\cdot 3\cdot 2} + \frac{1}{3\cdot 2\cdot 3\cdot 4} \right) \\
&= \frac{4!}{12} + \frac{4!}{36} + \frac{4!}{72} = 2 + \frac{2}{3} + \frac{1}{3} = 3.
\end{aligned}$$

- Note that we are adding non-integers to get an integer. A strong sign that this is a hard theorem!
- History:

2014: Naruse states his formula in a talk, omitting the proof. Supposedly his proof uses algebraic geometry, which is where the notion of excited diagrams originated.

2015–21: Morales, Pak and Panova write a series of four papers (#1, #2, #3, #4) giving several proofs, which are more combinatorial but still intricate and advanced.

2023: Grinberg, Korniichuk, Molokanov, Khomych (the latter three are high school students at the time of writing) find a proof that is completely elementary but very long (arXiv:2310.18275). This also proves a "weighted" generalization of the formula.

2024: Panova and Petrov publish two simpler proofs (arXiv:2409.17842) using a bit of complex analysis and lattice methods.

- Excited diagrams are hard to work with. Each proof (I think) starts out by re-encoding them as semistandard tableaux of shape μ satisfying certain conditions.

3. More about counting standard tableaux

3.1. Classical formulas

- Now we return to straight shapes $Y(\lambda)$.
- For any partition λ, let f^λ be the # of standard tableaux of shape λ.
- "$\lambda \vdash n$" is shorthand for "λ is a partition of n".

- **Classical theorems (Young 1928 and Littlewood?):** Let $n \in \mathbb{N}$. Then:

  (a) We have
  $$\sum_{\lambda \vdash n} \left(f^{\lambda}\right)^2 = n! = (\text{\# of all permutations } w \in S_n) .$$

  (b) We have
  $$\sum_{\lambda \vdash n} f^{\lambda} = \sum_{k=0}^{n} \frac{\dbinom{n}{2k}\dbinom{2k}{k} k!}{2^k} = (\text{\# of all involutions } w \in S_n) .$$

- Part **(a)** is Corollary 5.12.18 in the notes, where it is proved using Young symmetrizers and the center of $\mathbf{k}[S_n]$.

  Part **(b)** is Corollary 5.21.11 in the notes, where it is proved using a bespoke basis of $\mathbf{k}[S_n]$.

- But these are combinatorial identities, so can't we just prove them combinatorially?

- Yes! Two ways:

  - explicitly bijective via the *Robinson–Schensted–Knuth correspondence*;
  - recursive via "one-step bijections".

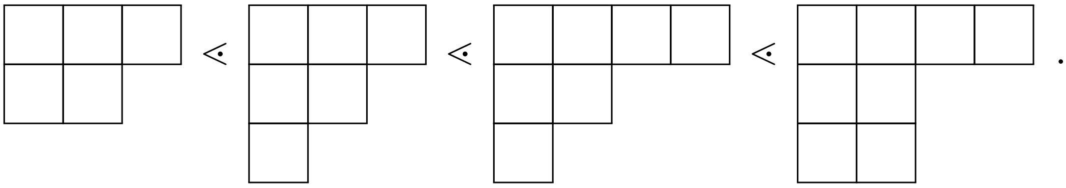
  Both are explained in Marc A. A. van Leeuwen, *The Robinson-Schensted and Schützenberger algorithms, an elementary approach*, 1996. In a deeper sense, they are equivalent, since the Robinson–Schensted–Knuth correspondence can be assembled from the "one-step bijections".

3.2. A proof sketch

- Let me outline the second proof (§1 of van Leeuwen's paper).

- If μ and λ are two partitions, then the notation "$\mu \lessdot \lambda$" will mean "$Y(\lambda) = Y(\mu) \cup \{\text{a single box not in } Y(\mu)\}$", or equivalently "$\mu \subseteq \lambda$ and $|\lambda| = |\mu| + 1$".

- **Example:**



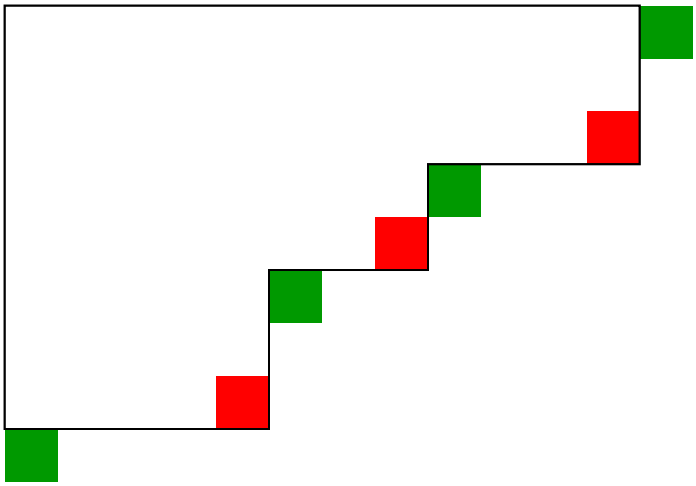
- **Lemma A:** For any partition λ, we have

$$\begin{aligned} & (\text{\# of partitions } \mu \text{ such that } \lambda \lessdot \mu) \\ & = (\text{\# of partitions } \mu \text{ such that } \mu \lessdot \lambda) + 1. \end{aligned}$$

Proof idea: We must prove that

$$\begin{aligned} & (\text{\# of ways to add a cell to } Y(\lambda) \text{ and get a partition}) \\ & = (\text{\# of ways to remove a cell from } Y(\lambda) \text{ and get a partition}) - 1. \end{aligned}$$

But the former and the latter ways "alternate" as you scan $Y(\lambda)$ from top to bottom:



(green = addable cells; red = removable cells).

- **Lemma B:** Given two partitions $\lambda \neq \nu$, we have

$$\begin{aligned} & (\text{\# of partitions } \mu \text{ such that } \lambda \lessdot \mu \text{ and } \nu \lessdot \mu) \\ & = (\text{\# of partitions } \mu \text{ such that } \mu \lessdot \lambda \text{ and } \mu \lessdot \nu). \end{aligned}$$

Proof idea: If λ and ν have the same size and differ in only two boxes, then this is saying $1 = 1$. Otherwise, it is saying $0 = 0$.

- **Lemma C:** For any nonempty partition λ, we have

$$\sum_{\mu \lessdot \lambda} f^\mu = f^\lambda.$$

Proof idea: This is our old friend

$$f^\lambda = f^{\lambda - e_1} + f^{\lambda - e_2} + \cdots + f^{\lambda - e_k}.$$

- **Lemma D:** For any partition λ, we have

$$\sum_{\lambda \lessdot \mu} f^\mu = \left(\left|\lambda\right| + 1\right) f^\lambda.$$

Proof idea: Combine Lemmas A, B and C and induct on $\left|\lambda\right|$. The induction step in a bit more detail:

$$\begin{aligned}
\sum_{\lambda \lessdot \mu} f^\mu &= \sum_{\lambda \lessdot \mu} \sum_{\nu \lessdot \mu} f^\nu && \text{(by Lemma C)} \\
&= f^\lambda + \sum_{\mu \lessdot \lambda} \sum_{\mu \lessdot \nu} f^\nu && \left(\begin{array}{c}\text{by comparing coefficients} \\ \text{in front of each } f^\nu, \\ \text{using Lemma A and Lemma B}\end{array}\right) \\
&= f^\lambda + \sum_{\mu \lessdot \lambda} \left(\left|\mu\right| + 1\right) f^\mu && \left(\begin{array}{c}\text{by induction hypothesis for } \mu, \\ \text{since } \left|\mu\right| = \left|\lambda\right| - 1\end{array}\right) \\
&= f^\lambda + \sum_{\mu \lessdot \lambda} \left|\lambda\right| f^\mu = f^\lambda + \left|\lambda\right| \sum_{\mu \lessdot \lambda} f^\mu \\
&= f^\lambda + \left|\lambda\right| f^\lambda && \text{(by Lemma C)} \\
&= \left(\left|\lambda\right| + 1\right) f^\lambda.
\end{aligned}$$

- Now, setting

$$\sigma_2(n) = \sum_{\lambda \vdash n} \left(f^\lambda\right)^2 \qquad \text{and} \qquad \sigma_1(n) = \sum_{\lambda \vdash n} f^\lambda,$$

we easily find the recursions

$$\begin{aligned}
\sigma_2(n) &= n \sigma_2(n-1) \qquad \text{and} \\
\sigma_1(n) &= \sigma_1(n-1) + (n-1) \sigma_1(n-2),
\end{aligned}$$

which lead to the explicit formulas $\sigma_2(n) = n!$ and $\sigma_1(n) =$ (# of all involutions $w \in S_n$), qed.

- The idea of this proof goes back to Rutherford, *Substitutional Analysis*, Edinburgh 1948, §15.