Math 235 Fall 2024, Lecture 9 stenogram: Enumerative combinatorics

website: `https://www.cip.ifi.lmu.de/~grinberg/t/24f`

1. Invariants and Monovariants

This was less of a proper lecture than a discussion session. See the lecture notes (Chapter 8) for details.

Exercise 1. A chunk of ice is floating in the sea. At each moment, a chunk can break into either 3 or 5 smaller chunks. Can we get precisely 100 chunks?

Solution. No.
Proof. The parity of the # of chunks is invariant.

Exercise 2. The numbers $1, 2, \ldots, 100$ are written in a row (in this order).

In a move, you can swap any two numbers at a distance of 2.

Can you end up with the same numbers in the reverse order $(100, 99, \ldots, 1)$?

Solution. No.
Proof. The set of all numbers in the even positions stays unchanged; so does the set of all numbers in the odd positions. Originally, 1 was in an odd position, but in the end we want it to move to an even position. This is clearly impossible.

Example 1.0.1. Let $n \geq 2$ be an integer. Consider n trees arranged in a circle, with one sparrow sitting on each tree.

Every minute, two of the n sparrows move: one moves to the next tree clockwise, and one moves to the next tree counterclockwise.

Is it possible that, after some time, all sparrows end up on the same tree? Answer this in dependence on n.

Solution. Yes if n is odd; no if n is even.
Proof. When n is odd, we can just pick a tree and have every sparrow move to that tree, making sure to always move them in pairs of opposite sparrows with respect to that tree.

When n is even, we can find an invariant:

$$\sum_{s \text{ is a sparrow}} (\text{the number of the tree that } s \text{ sits on}).$$

More precisely, this sum is not literally invariant, but it changes by increments of n, $-n$ or 0, so its remainder modulo n is an invariant. At the onset, this sum is $1+2+\cdots+n = \frac{n(n+1)}{2} \equiv \frac{n}{2} \not\equiv 0 \bmod n$, whereas our goal is to make it $n \cdot k \equiv 0 \bmod n$. So it is impossible.

Example 1.0.2. Let $n \geq 3$ and $m \geq 3$ be two integers. You have a rectangular $n \times m$-grid of lamps. Initially, all nm lamps are off. In a move, you can choose a row or a column of the grid, and flip all lamps in this row or column.

Can you, by such moves, obtain a state in which the four corner lamps are on while all the remaining lamps are off?

Solution. No.

Proof. Consider the first two lamps of the first two rows of the grid. These altogether four lamps have the property that at each point, an even number of them will be on (indeed, this number can only change by 2, -2 or 0 in any move). But initially, this number is 0, while our desired final state has it 1. So we cannot reach the final state.

Exercise 3. Let $n \in \mathbb{N}$. You start with an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers. In one move, you are allowed to pick two adjacent entries a_i and a_{i+1} that are out of order (i.e., satisfy $a_i > a_{i+1}$), and swap them. Prove that after at most $\binom{n}{2}$ such moves, the n-tuple will become weakly increasing.

Solution. An **inversion** of an n-tuple $(a_1, a_2, \ldots, a_n)$ means a pair $(i, j) \in [n]^2$ with $i < j$ and $a_i > a_j$. The total # of inversions of an n-tuple is always $\leq \binom{n}{2}$. In each of our moves, this # decreases by exactly 1. So this # is a (strict) monovariant.

Exercise 4. The numbers $1, 2, \ldots, 99$ are written in a row (in this order).

In a move, you can swap any two numbers at a distance of 2.

Can you end up with the same numbers in the reverse order $(99, 98, \ldots, 1)$?

Solution. Yes.

Proof. Reverse the process: Start with $99, 98, \ldots, 1$ and try to get $1, 2, \ldots, 99$.

To get it, make swaps of the form $\ldots abc \ldots \to \ldots cba \ldots$ whenever $a > c$.

This boils down to the preceding exercise, applied separately to the numbers in the even positions and to the numbers in the odd positions. So, after at most $\binom{50}{2} + \binom{49}{2}$ moves, the numbers in the even positions will be in increasing order, and so will be the numbers in the odd positions. Moreover, as we know, the sets of the numbers in either set of positions don't change. So the numbers in the even positions must be $2, 4, \ldots, 98$ from left to right, while the numbers in the odd positions must be $1, 3, \ldots, 99$ from left to right. So the total sequence of numbers is now $1, 2, 3, 4, \ldots, 99$, as desired.

Exercise 5. Let n and m be two positive integers. You have a rectangular $n \times m$-grid of lamps. A **line** shall mean a row or a column of the grid (so there are $n + m$ lines in total).

In a move, you can choose a line and flip all the lamps on this line.

Prove that – starting with an arbitrary state of the lamps – you can always find a sequence of moves after which each line has at least as many lamps turned on as it has lamps turned off.

Solution. Proceed as follows ("greedy algorithm"): Pick a line that has fewer on-lamps than off-lamps, and flip this line. Rinse and repeat.

Why will this terminate? Because the total # of on-lamps increases in each move.

Exercise 6. Let $n \geq 2$ be an integer. A country has n towns, the distances between which are distinct.

(a) You start in a town A_1. From there you travel to the town A_2 that is farthest away from A_1. From there you travel to the town A_3 that is farthest away from A_2. You continue travelling in this pattern.

Prove that if $A_3 \neq A_1$, then you never come back to A_1.

(b) Prove the same if "farthest away from" is replaced by "closest to".

Solution. **(a)** By construction,

$$|A_1 A_2| \leq |A_2 A_3| \leq |A_3 A_4| \leq \cdots .$$

If $A_1 \neq A_3$, then the first $\leq$ sign here is actually a $<$ sign, so we get $|A_1 A_2| < |A_i A_{i+1}|$ for all $i \geq 2$.

But if we come back to A_1, then $A_i = A_1$ for some $i > 1$, so that $A_{i+1} = A_2$ and therefore $|A_1 A_2| = |A_i A_{i+1}|$, contradiction.

(b) Analogous: just flip every inequality sign.