Math 235 Fall 2024, Lecture 5 stenogram: Sums and sequences

website: `https://www.cip.ifi.lmu.de/~grinberg/t/24f`

1. Sums and sequences (cont'd)

1.1. Guessing sequences

Often you have a sequence of numbers in front of you (e.g., given by a recursive definition) and you need to find its properties – e.g., an explicit formula for its entries (not always possible), or a proof that all its entries are integers or odd numbers or whatever, or just some patterns you suspect to exist. There is no one rule or one method for how to do this, but there are many techniques. Let us see some examples.

> **Exercise 1.** Let q and d be two numbers. Let $(x_0, x_1, x_2, \ldots)$ be a sequence of numbers that satisfies the recursive equation
>
> $$x_n = qx_{n-1} + d \qquad \text{for each } n \geq 1.$$
>
> (Such a sequence is called an **arithmetic-geometric progression**. Note that it is an arithmetic progression when $q = 1$ and a geometric progression when $d = 0$.)
>
> Find an explicit formula for x_n in terms of x_0, q, d.

Solution. Let us experiment with the first entries:

$$\begin{aligned}
x_0 &= x_0, \\
x_1 &= qx_0 + d, \\
x_2 &= qx_1 + d = q\left(qx_0 + d\right) + d = q^2 x_0 + \left(qd + d\right), \\
x_3 &= qx_2 + d = q\left(q^2 x_0 + \left(qd + d\right)\right) + d = q^3 x_0 + \left(q^2 d + qd + d\right), \\
&\ldots
\end{aligned}$$

We can venture a guess:

$$\begin{aligned}
x_n &= q^n x_0 + \left(q^{n-1} d + q^{n-2} d + \cdots + q^0 d\right) \\
&= q^n x_0 + \left(q^{n-1} + q^{n-2} + \cdots + q^0\right) d \\
&= q^n x_0 + \sum_{k=0}^{n-1} q^k d \\
&= q^n x_0 + \frac{q^n - 1}{q - 1} d \qquad \left(\text{for } q \neq 1\right).
\end{aligned}$$

This guess is correct and easy to prove by induction on n. ■

Exercise 2. Let $(a_1, a_2, a_3, \ldots)$ be a sequence of numbers defined recursively by $a_1 = 1$ and

$$a_1 + a_2 + \cdots + a_n = n^2 \cdot a_n \qquad \text{for all } n \geq 2.$$

Find an explicit formula for a_n.

Solution. We set

$$b_n := a_1 + a_2 + \cdots + a_n \qquad \text{for all } n \in \mathbb{N}$$

(in particular, $b_0 = (\text{empty sum}) = 0$). Then,

$$a_n = b_n - b_{n-1} \qquad \text{for all } n \geq 1.$$

So if we find an explicit formula for b_n, then we obtain an explicit formula for a_n.

Is b_n any simpler than a_n ? It is. In fact, our recurrence

$$a_1 + a_2 + \cdots + a_n = n^2 \cdot a_n$$

can be rewritten as

$$\begin{aligned} b_n &= n^2 \cdot (b_n - b_{n-1}), && \text{that is,} \\ b_n &= n^2 b_n - n^2 b_{n-1}, && \text{that is,} \\ n^2 b_{n-1} &= n^2 b_n - b_n, && \text{that is,} \\ n^2 b_{n-1} &= \left(n^2 - 1\right) b_n. \end{aligned}$$

Solving this for b_n, we find

$$b_n = \frac{n^2}{n^2 - 1} \cdot b_{n-1}.$$

So

$$\begin{aligned}
b_n &= \frac{n^2}{n^2-1} \cdot b_{n-1} \\
&= \frac{n^2}{n^2-1} \cdot \frac{(n-1)^2}{(n-1)^2-1} \cdot b_{n-2} \\
&= \frac{n^2}{n^2-1} \cdot \frac{(n-1)^2}{(n-1)^2-1} \cdot \frac{(n-2)^2}{(n-2)^2-1} \cdot b_{n-3} \\
&= \cdots \\
&= \frac{n^2}{n^2-1} \cdot \frac{(n-1)^2}{(n-1)^2-1} \cdot \frac{(n-2)^2}{(n-2)^2-1} \cdot \cdots \cdot \frac{2^2}{2^2-1} \cdot \underbrace{b_1}_{=a_1=1} \\
&= \frac{n^2}{n^2-1} \cdot \frac{(n-1)^2}{(n-1)^2-1} \cdot \frac{(n-2)^2}{(n-2)^2-1} \cdot \cdots \cdot \frac{2^2}{2^2-1} \\
&= \prod_{k=2}^{n} \frac{k^2}{k^2-1} = \prod_{k=2}^{n} \frac{k^2}{(k-1)(k+1)} \\
&= \frac{\left(\prod\limits_{k=2}^{n} k\right)^2}{\left(\prod\limits_{k=2}^{n} (k-1)\right)\left(\prod\limits_{k=2}^{n} (k+1)\right)} = \frac{(2 \cdot 3 \cdot \cdots \cdot n)^2}{(1 \cdot 2 \cdot \cdots \cdot (n-1)) \cdot (3 \cdot 4 \cdot \cdots \cdot (n+1))} \\
&= \frac{2n}{n+1}.
\end{aligned}$$

Alternatively, we can use the telescope trick:

$$\begin{aligned}
\prod_{k=2}^{n} \frac{k^2}{(k-1)(k+1)} &= \prod_{k=2}^{n} \left(\frac{k}{k-1} \Big/ \frac{k+1}{k}\right) = \frac{2}{1} \Big/ \frac{n+1}{n} \qquad \text{(by telescope)} \\
&= \frac{2n}{n+1}.
\end{aligned}$$

Now,

$$\begin{aligned}
a_n = b_n - b_{n-1} &= \frac{2n}{n+1} - \frac{2(n-1)}{(n-1)+1} = \frac{2n}{n+1} - \frac{2(n-1)}{n} \\
&= \frac{2}{n(n+1)}.
\end{aligned}$$

■

Exercise 3. Define a sequence $(a_1, a_2, a_3, \ldots)$ of rational numbers recursively by

$$a_1 = \frac{5}{2}, \qquad \text{and} \qquad a_n = a_{n-1}^2 - 2 \text{ for every } n \geq 2.$$

Find an explicit formula for a_n.

Solution. Again compute the first few values:

$$\begin{aligned}
a_1 &= \frac{5}{2}, \\
a_2 &= a_1^2 - 2 = \left(\frac{5}{2}\right)^2 - 2 = \frac{17}{4}, \\
a_3 &= a_2^2 - 2 = \left(\frac{17}{4}\right)^2 - 2 = \frac{257}{16}, \\
&\ldots.
\end{aligned}$$

We suspect

$$a_n = \frac{2^{2^n} + 1}{2^{2^{n-1}}} = 2^{2^{n-1}} + 2^{-2^{n-1}}.$$

Once guessed, this formula can be easily proved by induction: Assuming that $a_n = 2^{2^{n-1}} + 2^{-2^{n-1}}$, our recursion yields

$$\begin{aligned}
a_{n+1} = a_n^2 - 2 &= \left(2^{2^{n-1}} + 2^{-2^{n-1}}\right)^2 - 2 \\
&= \underbrace{\left(2^{2^{n-1}}\right)^2}_{=2^{2^n}} + 2 \cdot \underbrace{2^{2^{n-1}} \cdot 2^{-2^{n-1}}}_{=1} + \underbrace{\left(2^{-2^{n-1}}\right)^2}_{=2^{-2^n}} - 2 \\
&= 2^{2^n} + 2^{-2^n}.
\end{aligned}$$

How could we have obtained this answer without guessing it?

We observe that any number $y \neq 0$ satisfies

$$\left(y + y^{-1}\right)^2 - 2 = y^2 + 2\underbrace{yy^{-1}}_{=1} + y^{-2} - 2 = y^2 + y^{-2}.$$

So the expression $x^2 - 2$ can be simplified whenever x can be written in the form $y + y^{-1}$ for some y. Can every number x be written in this form? Essentially yes, if you allow y to be complex.

Thus, seeing the recursion $a_n = a_{n-1}^2 - 2$, we can make the substitution $a_n := b_n + b_n^{-1}$ and rewrite the recursion as $b_n + b_n^{-1} = b_{n-1}^2 + b_{n-1}^{-2}$. We can solve this recursion simply by setting $b_n = b_{n-1}^2$. And this latter recursion is easily solved:

$$b_n = b_{n-1}^2 = b_{n-2}^4 = b_{n-3}^8 = \cdots = b_1^{2^{n-1}}.$$

Thus, if we pick b_1 in such a way that $a_1 = b_1 + b_1^{-1}$, then we will have

$$a_n = b_n + b_n^{-1} = b_1^{2^{n-1}} + b_1^{-2^{n-1}}.$$

It remains to find b_1 in such a way that $a_1 = b_1 + b_1^{-1}$. In the case of our exercise, $a_1 = \frac{5}{2}$, so that we can take $b_1 = 2$. (If a_1 was 3, then b_1 would be $\frac{1}{2}\sqrt{5} + \frac{3}{2}$. If a_1 was 1, then b_1 would be $\frac{1}{2}i\sqrt{3} + \frac{1}{2}$, which is a complex number. The choice of $\frac{5}{2}$ in the exercise was meant to make the answer nicer.)

Another example in the notes (Exercise 4.6.4).

1.2. Periodicity

Periodicity is one of the simplest patterns that a sequence can have.

Definition 1.2.1. Let $u = (u_0, u_1, u_2, \ldots)$ be an infinite sequence (of any objects).

(a) A positive integer d is said to be a **period** of u if every $i \in \mathbb{N}$ satisfies $u_i = u_{i+d}$.

(b) The sequence u is said to be **periodic** if it has a period.

(c) Let d be a positive integer. The sequence u is said to be d**-periodic** if d is a period of u.

Example 1.2.2. A 1-periodic sequence is a sequence $(u_0, u_1, u_2, \ldots)$ that satisfies $u_i = u_{i+1}$ for all $i \in \mathbb{N}$, that is, that satisfies $u_0 = u_1 = u_2 = \cdots$. This is also known as a **constant sequence**.

Example 1.2.3. The sequence $\left((-1)^0, (-1)^1, (-1)^2, \ldots\right) = (1, -1, 1, -1, 1, -1, \ldots)$ is 2-periodic, and also d-periodic for any even positive integer d.

Example 1.2.4. For any positive integer n, the sequence

$$\begin{aligned}&(0\%n,\ 1\%n,\ 2\%n,\ \ldots)\\ &= (0, 1, 2, \ldots, n-1, 0, 1, 2, \ldots, n-1, 0, 1, 2, \ldots, n-1, \ldots)\end{aligned}$$

is n-periodic.

Example 1.2.5. Consider the sequence $(g_0, g_1, g_2, \ldots)$ defined like the Fibonacci sequence but with a little twist:

$$g_0 = 0, \qquad g_1 = 1,$$
$$g_n = g_{n-1} - g_{n-2} \qquad \text{for all } n \geq 2.$$

Then,
$$(g_0, g_1, g_2, \ldots) = (0, 1, 1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, \ldots)$$
is a 6-periodic sequence.

Example 1.2.6. The sequence $\left(0^0, 0^1, 0^2, 0^3, \ldots\right) = (1, 0, 0, 0, \ldots)$ is **not** periodic (since the entry 1 appears never again after its first position). It is, however, **eventually periodic** (i.e., it is periodic after removing a finite piece).

Theorem 1.2.7 (facts about periodic sequences)**.** Let $u = (u_0, u_1, u_2, \ldots)$ be an infinite sequence. Then:

(a) If a and b are two periods of u, then $a + b$ is a period of u.

(b) If a and b are two periods of u such that $a > b$, then $a - b$ is a period of u.

(c) If a is a period of u, then na is a period of u for any integer $n \geq 1$.

(d) If a is a period of u, and if $p, q \in \mathbb{N}$ satisfy $p \equiv q \operatorname{mod} a$, then $u_p = u_q$.

(e) Assume that u is periodic. Let m be the **smallest** period of u. Then, the periods of u are precisely the positive multiples of m.

Proof. Easy in this order; see the notes (Theorems 4.7.8 and 4.7.9). □

What works for sequences often works for functions as well. Periodicity is no exception:

Definition 1.2.8. Let $\mathbb{A}$ be either $\mathbb{R}$ or $\mathbb{R}_+ = \{\text{positive reals}\}$. Let S be any set. Let $u : \mathbb{A} \to S$ be a function.

(a) A positive real d is said to be a **period** of u if every $x \in \mathbb{A}$ satisfies $u(x) = u(x + d)$.

(b) The function u is said to be **periodic** if it has a period.

(c) Let d be a positive real. The function u is said to be d**-periodic** if d is a period of u.

Example 1.2.9. The trigonometric functions $\sin, \cos, \tan, \cot, \sec, \csc$ are 2π-periodic. Actually, $\tan$ and $\cot$ are π-periodic as well.

The above theorem ("facts about periodic sequences") has a partial analogue for periodic functions. Partial because not all of it extends to functions: A periodic function might fail to have a minimal period. This happens for constant functions (every positive real is a period) and for certain pathological discontinuous functions, but does not happen for non-constant continuous functions.

All I have said about periodicity is easy, but some of it is quite useful. Here an example:

Exercise 4. Let n be a positive integer. Prove that

$$\sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor = \lfloor nx \rfloor \qquad \text{for each } x \in \mathbb{R}.$$

Solution. Define the function $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) := \sum_{k=0}^{n-1} \left\lfloor x + \frac{k}{n} \right\rfloor - \lfloor nx \rfloor .$$

We must then show that f is constant 0 (that is, $f(x) = 0$ for all $x \in \mathbb{R}$).

But f is 1-periodic (since adding 1 to x causes all the floors $\left\lfloor x + \frac{k}{n} \right\rfloor$ to be incremented by 1, and causes the floor $\lfloor nx \rfloor$ to grow by n, and of course all these increments cancel). So all values of f are already taken on the interval $[0, 1)$. So we only need to show that f is constant 0 on this interval.

Why is f constant 0 on $[0, 1)$? Subdivide the interval $[0, 1)$ into n subintervals $\left[\frac{i}{n}, \frac{i+1}{n}\right)$ for $i \in \{0, 1, \ldots, n-1\}$. On each of these subintervals,

$$\begin{aligned} f(x) &= \sum_{k=0}^{n-1} \underbrace{\left\lfloor x + \frac{k}{n} \right\rfloor}_{\substack{=0 \text{ when } i<n-k; \\ =1 \text{ when } i \geq n-k}} - \underbrace{\lfloor nx \rfloor}_{=i} \\ &= (\text{sum of } i \text{ many 1s}) - i = i - i = 0. \end{aligned}$$

So f is 0 all over $[0, 1)$ and thus everywhere else as well. ■

What principle have we used in this proof? The principle that a d-periodic function is uniquely determined by its values on a given half-open interval of length d. Likewise, a d-periodic sequence is uniquely determined by any d consecutive entries.

1.3. Linear recurrences

The Fibonacci sequence is an instance of a wider class of sequences, which share most of its properties (not all – every once in a while you do come across a result that only holds for Fibonacci numbers). This general class are the **linearly recurrent sequences** (or, more precisely, **sequences that satisfy linear recurrences with constant coefficients**). This class also contains the arithmetic progressions, the geometric progressions, the arithmetic-geometric progressions, and many others. Their theory is interesting yet rather manageable, and we can even compute "explicit" formulas for their entries (like Binet's formula $f_n = \dfrac{\varphi^n - \psi^n}{\sqrt{5}}$), although these formulas involve irrationalities.

We will mostly discuss the most common subclass of linearly recurrent sequences: the ones with **two-term recurrences** (also known as **recurrences of the second degree**). These have the nicest properties and also are connected to many other things (trigonometry, ODEs, Chebyshev polynomials, ...).

Definition 1.3.1. Let a and b be two numbers. A sequence $(x_0, x_1, x_2, \ldots)$ of numbers will be called (a, b)**-recurrent** if every $n \geq 2$ satisfies

$$x_n = ax_{n-1} + bx_{n-2}.$$

Clearly, an (a, b)-recurrent sequence is uniquely determined by the four numbers a, b, x_0, x_1.

Example 1.3.2. The Fibonacci sequence $(f_0, f_1, f_2, \ldots) = (0, 1, 1, 2, 3, 5, 8, \ldots)$ is $(1, 1)$-recurrent. The Lucas sequence $(\ell_0, \ell_1, \ell_2, \ldots) = (2, 1, 3, 4, 7, \ldots)$ is also $(1, 1)$-recurrent.

Example 1.3.3. A sequence $(x_0, x_1, x_2, \ldots)$ is $(2, -1)$-recurrent if and only if it satisfies

$$x_n = 2x_{n-1} - x_{n-2} \qquad \text{for all } n \geq 2, \qquad \text{or equivalently}$$
$$x_n - x_{n-1} = x_{n-1} - x_{n-2} \qquad \text{for all } n \geq 2.$$

Thus, the $(2, -1)$-recurrent sequences are just the arithmetic progressions.

Example 1.3.4. Any geometric progression $(u, uq, uq^2, uq^3, \ldots)$ is $(q, 0)$-recurrent.

However, not every $(q, 0)$-recurrent sequence is a geometric progression! In fact, $(q, 0)$-recurrence only means that $x_n = qx_{n-1}$ for $n \geq 2$, not for $n \geq 1$, so it says nothing about x_0.

Example 1.3.5. What is a $(0, 1)$-recurrent sequence? A sequence $(x_0, x_1, x_2, \ldots)$ satisfying $x_n = x_{n-2}$ for all $n \geq 2$. That is, a 2-periodic sequence.

Example 1.3.6. Every $(1, -1)$-recurrent sequence $(x_0, x_1, x_2, \ldots)$ is 6-periodic:

$$\begin{aligned} x_{n+6} &= x_{n+5} - x_{n+4} = (x_{n+4} - x_{n+3}) - x_{n+4} = -x_{n+3} \\ &= -(x_{n+2} - x_{n+1}) = x_{n+1} - x_{n+2} = x_{n+1} - (x_{n+1} - x_n) = x_n. \end{aligned}$$

However, not every 6-periodic sequence is $(1, -1)$-recurrent.

Example 1.3.7. Let α be any angle. Then, the sequences

$$(\sin(0\alpha), \sin(1\alpha), \sin(2\alpha), \ldots) \qquad \text{and}$$
$$(\cos(0\alpha), \cos(1\alpha), \cos(2\alpha), \ldots)$$

are $(2\cos\alpha, -1)$-recurrent. More generally, the sequence

$$(\sin(\beta+0\alpha), \sin(\beta+1\alpha), \sin(\beta+2\alpha), \ldots)$$

is $(2\cos\alpha, -1)$-recurrent for any angle β.

See the notes for a proof.

What can we say about an arbitrary (a,b)-recurrent sequence? We can try to find an explicit formula:

Exercise 5. Let a and b be two numbers, and let $(x_0, x_1, x_2, \ldots)$ be an (a,b)-recurrent sequence. Is there an explicit formula for x_n in terms of a, b, x_0, x_1, similar to Binet's formula for Fibonacci numbers?

Solution. (See the notes for all details.) We try to imitate Binet's formula

$$f_n = \frac{1}{\sqrt{5}}\varphi^n - \frac{1}{\sqrt{5}}\psi^n.$$

We expect a formula of the form

$$x_n = \gamma\lambda^n + \delta\mu^n,$$

where $\gamma, \lambda, \delta, \mu$ are constants. This is an example of an **ansatz** (an incomplete guess, which has to be completed by determining the parameters that appear in it). An ansatz may succeed or fail, but let's run with this one and see if it perhaps succeeds.

If $x_n = \gamma\lambda^n + \delta\mu^n$ is to be true for all n, then the recursion

$$x_n = ax_{n-1} + bx_{n-2}$$

becomes

$$\gamma\lambda^n + \delta\mu^n = a\left(\gamma\lambda^{n-1} + \delta\mu^{n-1}\right) + b\left(\gamma\lambda^{n-2} + \delta\mu^{n-2}\right).$$

So this latter equality must hold for all $n \geq 2$. We suspect that the "λ-part" and the "μ-part" of this equality hold separately, i.e., that we have

$$\gamma\lambda^n = a\gamma\lambda^{n-1} + b\gamma\lambda^{n-2} \qquad \text{and}$$
$$\delta\mu^n = a\delta\mu^{n-1} + b\delta\mu^{n-2}.$$

If this is to hold, then we can divide the former equality by $\gamma\lambda^{n-2}$ and the latter by $\delta\mu^{n-2}$, simplifying both equalities to

$$\lambda^2 = a\lambda + b \qquad \text{and}$$
$$\mu^2 = a\mu + b.$$

So λ and μ should be roots of the quadratic polynomial $X^2 - aX - b$. Moreover, they should be the two distinct roots of this quadratic polynomial. By the quadratic formula, we know that the roots of this polynomial are

$$\frac{a - \sqrt{a^2 + 4b}}{2}, \qquad \frac{a + \sqrt{a^2 + 4b}}{2}.$$

So we take

$$\lambda = \frac{a + \sqrt{a^2 + 4b}}{2}, \qquad \mu = \frac{a - \sqrt{a^2 + 4b}}{2}.$$

It remains to find γ and δ. For this purpose, we consider the equality

$$x_n = \gamma\lambda^n + \delta\mu^n$$

for $n = 0$ and for $n = 1$. That is,

$$x_0 = \gamma\lambda^0 + \delta\mu^0 = \gamma + \delta,$$
$$x_1 = \gamma\lambda^1 + \delta\mu^1 = \gamma\lambda + \delta\mu.$$

Knowing λ, μ, x_0, x_1, we can view this as a system of two linear equations in γ, δ, which we can solve to get

$$\gamma = \frac{x_1 - \mu x_0}{\lambda - \mu}, \qquad \delta = \frac{\lambda x_0 - x_1}{\lambda - \mu}.$$

So our ansatz becomes the explicit formula

$$x_n = \frac{x_1 - \mu x_0}{\lambda - \mu}\lambda^n + \frac{\lambda x_0 - x_1}{\lambda - \mu}\mu^n,$$

where

$$\lambda = \frac{a + \sqrt{a^2 + 4b}}{2}, \qquad \mu = \frac{a - \sqrt{a^2 + 4b}}{2}.$$

This formula is a generalization of Binet's formula (which is obtained for $a = 1$ and $b = 1$ and $x_0 = 0$ and $x_1 = 1$). But there is a little problem: $\lambda - \mu$ can be 0, in which case the denominators in γ and δ become 0. Next time we will see how to handle this case. (This is the case when $a^2 + 4b = 0$.)