Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2024: Homework 9

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Please solve 4 of the 8 exercises!
Deadline: December 11, 2024

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$n! = \prod_{k=1}^{n} \operatorname{lcm}\left(1, 2, \ldots, \left\lfloor \frac{n}{k} \right\rfloor\right).$$

1.2 Solution

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2 Exercise 2

2.1 Problem

Let p be a prime. Let $k \in \mathbb{N}$ be such that k is not a positive multiple of $p-1$. Prove that

$$\sum_{i=0}^{p-1} i^k \equiv 0 \mod p.$$

2.2 Solution

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3 Exercise 3

3.1 Problem

Let a, b, c, d be four integers such that $ab = cd$. Prove that there exist four integers x, y, z, w such that

$$a = xy, \qquad b = zw, \qquad c = xz, \qquad d = yw.$$

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=1}^{n} \phi(k) \left\lfloor \frac{n}{k} \right\rfloor = \frac{n(n+1)}{2}.$$

4.2 Solution

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5 Exercise 5

5.1 Problem

Let n be a positive integer. Show that

$$\sum_{j=1}^{n} \operatorname{lcm}(j,n) = \frac{n}{2}\left(1 + \sum_{d\mid n} d\phi(d)\right).$$

(Here, as usual, the summation sign $\sum_{d\mid n}$ means a sum over all positive divisors d of n.)

5.2 Solution

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6 Exercise 6

6.1 Problem

Find all pairs (x, y) of nonnegative integers such that $(1 - x + y)^2 = xy$.

6.2 Hint

Start by showing that x and y must be coprime.

6.3 Solution

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7 Exercise 7

7.1 Problem

Let p and q be two distinct primes.

(a) Show that $p^q + q^p \equiv p + q \mod pq$.

(b) Assume that $p > 2$ and $q > 2$. Show that the integer $\left\lfloor \frac{p^q + q^p}{pq} \right\rfloor$ is even.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let p be a prime, and let $a, b \in \{0, 1, \ldots, p-1\}$. Prove that

$$\binom{a}{b} \equiv (-1)^{a-b} \binom{p-1-b}{p-1-a} \mod p.$$

8.2 Solution

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References