Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2024: Homework 7

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Please solve 3 of the 8 exercises!
Deadline: November 27, 2024

Recall that $[k] := \{1, 2, \ldots, k\}$ for each $k \in \mathbb{N}$.

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1 Exercise 1

1.1 Problem

Let n be a positive integer. A subset S of $[n]$ is said to be *cyclically lacunar* if it is lacunar[1] and also satisfies $\{1, n\} \not\subseteq S$ (that is, $1 \notin S$ or $n \notin S$). (In other words, if we arrange the numbers $1, 2, \ldots, n$ in this order on a circle, then a cyclically lacunar subset is a subset that contains no two adjacent numbers.)

Let $k \in \{0, 1, \ldots, n-1\}$. Prove that the # of cyclically lacunar k-element subsets of $[n]$ is $\dfrac{n}{n-k}\dbinom{n-k}{k}$.

1.2 Remark

Compare with the formula $\dbinom{n-k}{k}$ for the # of lacunar k-element subsets of $[n-1]$. This formula appeared in the proof of Proposition 4.3.20 in the notes, and can be used without proof.

[1] Recall that a set of integers is said to be *lacunar* if it contains no two consecutive integers.

1.3 Solution

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2 Exercise 2

2.1 Problem

Let $n \in \mathbb{N}$.

(a) Find an explicit formula for

$$\sum_{k=0}^{n} \frac{\dbinom{n}{k}^2}{\dbinom{2n}{2k}}.$$

(b) Find an explicit formula for

$$\sum_{k=0}^{n} \binom{n}{k} k \wr (n-k) \wr,$$

where $m \wr$ denotes the product of the first m odd positive integers (that is, $1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2m-1)$).

2.2 Solution

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3 Exercise 3

3.1 Problem

Let $N \in \mathbb{N}$. Let $(f_0, f_1, \ldots, f_N)$ and $(g_0, g_1, \ldots, g_N)$ be two N-tuples of numbers. Assume that

$$g_n = \sum_{i=n}^{N} (-1)^i \binom{i}{n} f_i \qquad \text{for every } n \in \{0, 1, \ldots, N\}.$$

Prove that

$$f_n = \sum_{i=n}^{N} (-1)^i \binom{i}{n} g_i \qquad \text{for every } n \in \{0, 1, \ldots, N\}.$$

3.2 Remark

This is a "mirror version" of binomial inversion (Exercise 7.7.8 in the notes).

3.3 Solution

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4 Exercise 4

4.1 Problem

Let n be a positive integer.

(a) A number $i \in \{1, 2, \ldots, n-1\}$ is called a *descent* of a permutation $w \in S_n$ if it satisfies $w(i) > w(i+1)$. Compute the average number of descents in a (uniformly random) permutation $w \in S_n$. (That is, compute $\dfrac{1}{n!} \sum\limits_{w \in S_n} (\# \text{ of descents of } w)$.)

(b) A number $i \in \{2, 3, \ldots, n-1\}$ is called a *peak* of a permutation $w \in S_n$ if it satisfies $w(i-1) < w(i) > w(i+1)$. Compute the average number of peaks in a (uniformly random) permutation $w \in S_n$.

4.2 Hint

Recall linearity of expectation.

4.3 Solution

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5 Exercise 5

5.1 Problem

Let $n, m \in \mathbb{N}$.

A function $f : [n] \to [m]$ is said to be *Smirnov* if $f(i) \neq f(i+1)$ for all $i \in [n-1]$. (In other words, f is Smirnov if and only if the n-tuple $(f(1), f(2), \ldots, f(n))$ is Smirnov, according to Definition 7.3.11 in the notes.)

For any function $f : [n] \to [m]$, we let $\operatorname{Fix} f := \{x \in [n] \mid f(x) = x\}$. (This is a slight extension of Definition 7.4.5 in the notes.)

Prove that

$$\sum_{\substack{f:[n]\to[m] \\ \text{is Smirnov}}} |\operatorname{Fix} f| = \min\{n, m\} \cdot (m-1)^{n-1}.$$

5.2 Solution

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6 Exercise 6

6.1 Problem

If a and b are two integers, then $[a, b]$ shall denote the set $\{a, a+1, a+2, \ldots, b\} \subseteq \mathbb{Z}$ (which is empty if $a > b$).

A function $f : [a, b] \to [c, d]$ is said to be *weakly increasing* if $f(a) \leq f(a+1) \leq \cdots \leq f(b)$. (This is understood to be automatically satisfied when $a \geq b$.)

(a) Let $n, m \in \mathbb{N}$. Prove that the # of weakly increasing functions $f : [1, n] \to [1, m]$ equals $\dbinom{n+m-1}{n}$.

(b) Let n be a positive integer. Prove that

$$\sum_{\substack{f:[1,n]\to[1,n] \\ \text{is weakly increasing}}} |\operatorname{Fix} f| = 4^{n-1},$$

where $\operatorname{Fix} f := \{x \in [1, n] \mid f(x) = x\}$.

6.2 Solution

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7 Exercise 7

7.1 Problem

If S is a finite set of integers, then $\operatorname{sum} S$ shall mean the sum of all elements of S (that is, $\sum\limits_{s \in S} s$).

For any $n \in \mathbb{N}$, we let

$$a_n := (\# \text{ of subsets } S \text{ of } [n] \text{ satisfying } 2 \mid \operatorname{sum} S) \qquad \text{and}$$
$$b_n := (\# \text{ of subsets } S \text{ of } [n] \text{ satisfying } 3 \mid \operatorname{sum} S).$$

(For instance, $b_4 = 6$, since the subsets of $[4]$ whose sum of elements is divisible by 3 are $\varnothing$, $\{3\}$, $\{1, 2\}$, $\{2, 4\}$, $\{1, 2, 3\}$ and $\{2, 3, 4\}$.)

(a) Prove that $a_n = 2^{n-1}$ for each $n \geq 1$.

(b) Prove that $b_n = 2b_{n-3} + 2^{n-2}$ for any $n \geq 3$.

(c) Prove that $b_n = 2b_{n-1} - [3 \mid n-1] \cdot 2^{(n-1)/3}$ for any $n \geq 1$. (Here, $[3 \mid n-1]$ denotes the truth value of $3 \mid n-1$ – that is, the number 1 if $3 \mid n-1$ and the number 0 otherwise.)

7.2 Remark

The sequence $(b_0, b_1, b_2, \ldots)$ is Sequence A068010 in the OEIS. An explicit formula – with a case distinction based on $n\%3$ – could be derived from part **(c)**, but would not be particularly aesthetic.

7.3 Solution

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8 Exercise 8

8.1 Problem

Let $n \in \mathbb{N}$ be even.

An n-tuple $a = (a_1, a_2, \ldots, a_n) \in \{1, -1\}^n$ consisting of 1's and -1's is said to be *balanced* if $a_1 + a_2 + \cdots + a_n = 0$ (that is, if there are equally many 1's and -1's in the n-tuple).

(a) Prove that the # of balanced n-tuples $a \in \{1, -1\}^n$ is $\dbinom{n}{n/2}$.

If $a = (a_1, a_2, \ldots, a_n) \in \{1, -1\}^n$ is a balanced n-tuple, then a *break-even point* of a means a number $k \in \{0, 1, \ldots, n\}$ such that $a_1 + a_2 + \cdots + a_k = 0$. Note that 0 and n always are break-even points of a, but there may be more (e.g., the number 4 is a break-even point of $(1, 1, -1, -1, 1, -1)$).

(b) Prove that any break-even point of a balanced n-tuple $a \in \{1, -1\}^n$ is even.

(c) Prove that the average number of break-even points of a (uniformly random) balanced n-tuple $a \in \{1, -1\}^n$ is $2^n / \dbinom{n}{n/2}$. In other words, prove that

$$\sum_{\substack{a \in \{1,-1\}^n \\ \text{is balanced}}} (\text{\# of break-even points of } a) = 2^n.$$

8.2 Solution

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References