Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2024: Homework 6

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Please solve 5 of the 10 exercises!
Deadline: November 20, 2024

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1 Exercise 1

1.1 Problem

Let $(a_0, a_1, a_2, \ldots)$ be a sequence of positive integers such that

$$a_n = \left(a_{n-1}^2 \% a_{n-2}\right) + 1 \qquad \text{for each } n \geq 2.$$

Prove that this sequence is eventually 2-periodic (i.e., there exists some $m \in \mathbb{N}$ such that the subsequence $(a_m, a_{m+1}, a_{m+2}, \ldots)$ is 2-periodic).

1.2 Solution

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2 Exercise 2

2.1 Problem

Find all pairs (x, y) of nonnegative integers such that $|x^2 - xy - y^2| = 5$. (Recall the Lucas sequence from Example 4.9.3 in the notes.)

2.2 Solution

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3 Exercise 3

3.1 Problem

Let n be a positive integer. Let a be any integer. Prove that there exist $i \in \mathbb{N}$ and $x \in \mathbb{Z}$ such that $a^{i+1}x \equiv a^i \mod n$.

3.2 Remark

This x can be viewed as a weak version of a modular inverse of a modulo n. An actual modular inverse would satisfy the congruence $ax \equiv 1 \mod n$ (that is, $a^{i+1}x \equiv a^i \mod n$ for $i = 0$), but it only exists when $a \perp n$ (see Theorem 3.5.9 in the notes), whereas the weak version always exists according to this exercise.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let n be a positive integer. Let $x_1, x_2, \ldots, x_{n+2}$ be $n + 2$ integers. Prove that there exist two distinct elements i and j of $\{1, 2, \ldots, n + 2\}$ such that $x_i - x_j$ or $x_i + x_j$ (or both) is divisible by $2n$.

4.2 Solution

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5 Exercise 5

5.1 Problem

Inside a regular hexagon with sidelength 1, you have marked 7 points. Show that there are two marked points whose distance to each other is at most 1.

5.2 Solution

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6 Exercise 6

6.1 Problem

Let $a \in \mathbb{Z}$ be such that $a \equiv 1 \mod 3$. Let k be a positive integer.

(a) Prove that there exists a unique $x \in \left\{0, 1, \ldots, 3^k - 1\right\}$ such that $x \equiv 1 \mod 3$ and $x^2 \equiv a \mod 3^k$.

(b) Prove that for each $i \in \mathbb{N}$, there exists a unique $x_i \in \left\{0, 1, \ldots, 3^k - 1\right\}$ such that $x_i \equiv 1 \mod 3$ and $x_i^{2^i} \equiv a \mod 3^k$.

(c) Consider the sequence $(x_0, x_1, x_2, \ldots)$ of these unique x_i's. Is this sequence periodic?

6.2 Remark

The x in part **(a)** can be regarded as the "canonical" square root of a modulo 3^k. In general, square roots modulo n neither necessarily exist nor are always unique, just like square roots of real numbers; thus the claim of part **(a)** is rather remarkable.

6.3 Solution

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7 Exercise 7

7.1 Problem

Let $a_1, a_2, \ldots, a_m$ be m integers, and let n be a positive integer such that $n < 2^m$. Prove that we can find m numbers $x_1, x_2, \ldots, x_m \in \{0, 1, -1\}$ such that not all these numbers $x_1, x_2, \ldots, x_m$ are zero and such that $n \mid \sum\limits_{i=1}^{m} x_i a_i$. (In other words, prove that we can make the sum $a_1 + a_2 + \cdots + a_m$ divisible by n if we are allowed to throw some (not all) of the addends away and replace some (possibly none, possibly all) of the remaining addends by their negatives.)

[**Example:** Let $m = 4$ and $n = 13$ and $(a_1, a_2, \ldots, a_m) = (4, 7, 19, 40)$. Then, the four numbers $0, -1, 1, 1$ are m numbers $x_1, x_2, \ldots, x_m$ that satisfy $n \mid \sum_{i=1}^{m} x_i a_i$, since $13 \mid 0 \cdot 4 + (-1) \cdot 7 + 1 \cdot 19 + 1 \cdot 40$.]

7.2 Solution

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8 Exercise 8

8.1 Problem

Let $a_1, a_2, \ldots, a_k$ be k real numbers. Let $(x_0, x_1, x_2, \ldots)$ be a sequence of real numbers that is $(a_1, a_2, \ldots, a_k)$-recurrent (see Definition 4.9.24 in the notes). Assume that this sequence has only finitely many distinct entries (i.e., the set $\{x_0, x_1, x_2, \ldots\}$ is finite). Show the following:

(a) The sequence $(x_0, x_1, x_2, \ldots)$ is eventually periodic (i.e., there exists some $m \in \mathbb{N}$ such that the sequence $(x_m, x_{m+1}, x_{m+2}, \ldots)$ is periodic).

(b) If $a_k \neq 0$, then the sequence $(x_0, x_1, x_2, \ldots)$ is periodic.

8.2 Solution

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9 Exercise 9

9.1 Problem

Let $n > 1$. Let A be an $n \times n$-matrix whose entries are the n^2 integers $1, 2, \ldots, n^2$ in some arbitrary order.

(a) Prove that we can find two entries i and j of A that lie in the same row and satisfy $\dfrac{n-1}{n} \leq \dfrac{i}{j} < 1$.

(b) Prove that we can find two entries i and j of A that lie in the same row or the same column and satisfy $\dfrac{n}{n+1} \leq \dfrac{i}{j} < 1$.

9.2 Solution

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10 Exercise 10

10.1 Problem

Let a and n be two integers such that $n > 0$. Let u and v be two positive integers such that $uv > n$ and $v > 1$. Prove that there exist two integers x and y with $|x| < u$ and $0 < y < v$ and $ay \equiv x \mod n$.

10.2 Solution

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References