Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2024: Homework 5

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Please solve 4 of the 8 exercises!
Deadline: November 6, 2024

1 Exercise 1

1.1 Problem

(a) Does Exercise 5.2.2 in the notes[1] remain valid if we replace "real axis" by "unit circle", replace "intervals" by "arcs", and replace "reals" by "points"?

(b) Does it help if we weaken the claim to "there exist three points a, b, c such that each of the arcs $I_1, I_2, \ldots, I_n$ contains at least one of a, b, c"?

(c) Find an analogous variant of Exercise 5.4.8 in the notes[2] for arcs on the unit circle instead of intervals on the real axis.

[1] Let n be a positive integer. Let $I_1, I_2, \ldots, I_n$ be n nonempty finite closed intervals on the real axis. Assume that for any three distinct elements $i, j, k \in \{1, 2, \ldots, n\}$, at least two of the three intervals I_i, I_j, I_k intersect. Prove that there exist two reals a and b such that each of the intervals $I_1, I_2, \ldots, I_n$ contains at least one of a and b.

[2] Let n and k be positive integers with $k \geq 2$. Let $I_1, I_2, \ldots, I_n$ be n nonempty finite closed intervals on the real axis. Assume that for any k distinct elements $i_1, i_2, \ldots, i_k \in \{1, 2, \ldots, n\}$, at least two of the k intervals $I_{i_1}, I_{i_2}, \ldots, I_{i_k}$ intersect. Prove that there exist $k-1$ reals $a_1, a_2, \ldots, a_{k-1}$ such that each of the intervals $I_1, I_2, \ldots, I_n$ contains at least one of $a_1, a_2, \ldots, a_{k-1}$.

1.2 Solution

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2 Exercise 2

2.1 Problem

Find all 5-tuples $(x_1, x_2, x_3, x_4, x_5)$ of nonnegative real numbers that satisfy the system of equations

$$\begin{cases} x_5 + x_2 = x_1^2; \\ x_1 + x_3 = x_2^2; \\ x_2 + x_4 = x_3^2; \\ x_3 + x_5 = x_4^2; \\ x_4 + x_1 = x_5^2 \end{cases}$$

(that is, $x_{i-1} + x_{i+1} = x_i^2$ for all $i \in \{1, 2, 3, 4, 5\}$, where $x_0 := x_5$ and $x_6 := x_1$).

2.2 Remark

Without the "nonnegative" requirement, this system has some messy solutions that cannot be written in any reasonable form.

2.3 Solution

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. Let A be an $n \times n$-matrix whose n^2 entries are distinct real numbers. Each row of A contains a largest entry; this entry will be called a *rowmax*. Each column of A contains a largest entry; this entry will be called a *colmax*.

(a) Assume that all the n rowmaxes lie in distinct columns, and that all the n colmaxes lie in distinct rows. Prove that the rowmaxes are exactly the colmaxes.

(b) Find a matrix A that does satisfy these assumptions.

3.2 Solution

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4 Exercise 4

4.1 Problem

Consider an $n \times m$-matrix A whose entries are 0's and 1's. Assume that each row of A contains at least one 0, whereas each column of A contains at least one 1. Let $A_{r,c}$ denote the entry of A in the r-th row and the c-th column.

Prove that there exist two distinct $r, s \in \{1, 2, \ldots, n\}$ and two distinct $c, d \in \{1, 2, \ldots, m\}$ such that $A_{r,c} = A_{s,d} = 1$ but $A_{r,d} = A_{s,c} = 0$.

4.2 Solution

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5 Exercise 5

5.1 Problem

For each $n \in \mathbb{N}$, define an integer a_n as follows: We know from Theorem 5.2.1 in the notes that there is a unique finite subset T of $\mathbb{N}$ such that $n = \sum_{t \in T} 2^t$. Consider this subset T, and set

$$a_n := \sum_{t \in T} \left(2^{t+1} - 1\right).$$

For instance, for $n = 10$, we have $T = \{3, 1\}$ (since $n = 10 = 2^3 + 2^1$) and thus $a_n = \sum_{t \in \{3,1\}} (2^{t+1} - 1) = (2^4 - 1) + (2^2 - 1) = 18$.

(a) Prove that $0 = a_0 < a_1 < a_2 < \cdots$.

(b) Prove that $a_n = a_{\lfloor n/2 \rfloor} + n$ for each $n \in \mathbb{N}$.

5.2 Solution

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6 Exercise 6

6.1 Problem

Let a, b, c be three integers. Prove that the number $a^2 + b^2 + c^2$ cannot be written in the form $4^k (8m + 7)$ with $k \in \mathbb{N}$ and $m \in \mathbb{Z}$.

6.2 Remark

This is part of the Legendre three-squares theorem. The other (much harder) part is the converse: Every nonnegative integer that is not of the form $4^k (8m+7)$ with $k \in \mathbb{N}$ and $m \in \mathbb{Z}$ can be written as a sum of three perfect squares. The proof of this can be found in some of the more advanced texts on elementary number theory, such as [UspHea39, §XIII.10].

6.3 Solution

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7 Exercise 7

7.1 Problem

Let $(a_0, a_1, a_2, \ldots)$ be a sequence of positive integers such that

$$a_n = \left(a_{n-2}^2 \% a_{n-1}\right) + 1 \qquad \text{for each } n \geq 2.$$

Prove that there exists some $m \in \mathbb{N}$ such that $a_m = a_{m+1} = a_{m+2} = \cdots = 1$.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let u and n be integers such that $n > 0$. Prove the following:

(a) We have $u - 1 \mid \dbinom{un-1}{n}$.

(b) If u is odd, then $\sum\limits_{k=0}^{n} \dbinom{un}{k}$ is even.

8.2 Solution

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References

[UspHea39] J. V. Uspensky, M. A. Heaslet, *Elementary Number Theory*, 1st edition, McGraw-Hill 1939.
See `https://projecteuclid.org/euclid.bams/1183502492` for some corrections.