Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2024: Homework 4

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Please solve 4 of the 8 exercises!
Deadline: October 30, 2024

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1 Exercise 1

1.1 Problem

Prove yet another formula for the Fibonacci numbers $f_0, f_1, f_2, \ldots$, namely the formula

$$f_{n+2} = \sum_{k=0}^{n} \binom{\lfloor (n+k)/2 \rfloor}{k} \qquad \text{for all } n \geq 0.$$

1.2 Solution

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2 Exercise 2

2.1 Problem

Consider the sequence $(a_0, a_1, a_2, \ldots)$ of integers defined recursively by

$$a_0 = 0, \qquad a_1 = 1, \qquad \text{and}$$
$$a_n = 1 + a_{n-1}a_{n-2} \qquad \text{for each integer } n \geq 2.$$

(This was studied in Exercise 4.8.1 of the notes.) Prove that $4 \nmid a_n$ for all positive integers n.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let $(a_0, a_1, a_2, \ldots)$ be a sequence of reals such that all positive integers k satisfy

$$a_k = 2a_{k-1} + 2^k. \tag{1}$$

Set $c := a_0$. Find an explicit formula for a_n in terms of c.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $(x_0, x_1, x_2, \ldots)$ be a sequence of real numbers defined recursively by

$$x_0 = 0 \qquad \text{and} \qquad x_1 = 1 \qquad \text{and} \qquad x_n = x_{n-1} + \frac{x_{n-2}}{n-1} \text{ for all } n \geq 2.$$

Express x_n as a finite sum.

4.2 Remark

With a correct answer and some known results from analysis, it is not hard to see that $\lim\limits_{n\to\infty} \dfrac{n}{x_n} = e \approx 2.718$.

4.3 Solution

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5 Exercise 5

5.1 Problem

Let n be a positive integer. Prove that

$$\sum_{k\in\{1,2,\ldots,2n\} \text{ is odd}} \left\lfloor \frac{k^3}{2n} \right\rfloor = n\left(n-1\right)\left(n+1\right).$$

5.2 Solution

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6 Exercise 6

6.1 Problem

Define a sequence $(a_0, a_1, a_2, \ldots)$ of real numbers recursively by

$$a_0 = 1 \qquad \text{and} \qquad a_n = -\sum_{k=1}^{n} \frac{a_{n-k}}{k!} \text{ for each } n \geq 1.$$

Find an explicit formula for a_n.

6.2 Solution

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7 Exercise 7

7.1 Problem

(a) A *linear fractional function* means a function $f : \mathbb{C} \setminus \{t\} \to \mathbb{C}$ given by the formula

$$f\left(x\right) = \frac{ax+b}{cx+d} \qquad \text{for all } x \in \mathbb{C} \setminus \{t\},$$

where $a, b, c, d \in \mathbb{C}$ are four constants. Here, t is the constant $-d/c$, which must be excluded from the domain in order to ensure that the denominators are nonzero.

Show that a composition of two linear fractional functions is again linear fractional (up to the fact that its domain might be a bit smaller in order to avoid zero denominators).

(b) Let $f : \mathbb{C} \setminus \{-2\} \to \mathbb{C}$ be the linear fractional function given by

$$f(x) = \frac{x+1}{x+2}.$$

For each $n \in \mathbb{N}$, let $f^{\circ n}$ be the function $\underbrace{f \circ f \circ \cdots \circ f}_{n \text{ times}}$ (obtained by applying f successively for a total of n times). Find an explicit formula for $f^{\circ n}(x)$ in terms of numbers known to us.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let x be a positive real number. Prove that

$$\sqrt{x+1} = 1 + \cfrac{x}{2 + \cfrac{x}{2 + \cfrac{x}{2 + \cfrac{x}{\ddots}}}}.$$

Here, the infinite continued fraction should be understood as in Exercise 2.4.2 of the notes; all the numbers in front of the + signs are 2's except for the very first one.

8.2 Solution

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References