Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2024: Homework 3

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Please solve 4 of the 8 exercises!
Deadline: October 23, 2024

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1 Exercise 1

1.1 Problem

Let $n \in \mathbb{N}$ be **odd**. Let $a_1, a_2, \ldots, a_n$ be n **odd** integers. Prove that

$$\sum_{1 \leq i < j \leq n} a_i a_j \equiv \binom{n}{2} \mod 4.$$

1.2 Solution

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2 Exercise 2

2.1 Problem

Find a formula for $\sum\limits_{i=0}^{n} i \dbinom{i}{k}$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N}$.

2.2 Solution

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3 Exercise 3

3.1 Problem

A *triangular number* means a number of the form $k\left(k+1\right)/2 = \dbinom{k+1}{2}$, where k is an integer. A *perfect square* means a number of the form k^2, where k is an integer.

Let n be an integer. Prove that n can be written as a sum of two triangular numbers if and only if $4n+1$ can be written as a sum of two perfect squares.

3.2 Remark

There is a celebrated result by Lagrange saying that a positive integer n can be written as a sum of two perfect squares if and only if each prime factor p satisfying $p \equiv 3 \mod 4$ appears with an even exponent in the prime factorization of n. A further result by Jacobi even gives a formula for the number of ways to write n as such a sum. However, none of this is needed for this problem.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let n and k be positive integers satisfying $k < 2^n$. Prove that the binomial coefficient $\dbinom{2^n - k - 1}{k}$ is even.

4.2 Solution

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5 Exercise 5

5.1 Problem

For each finite set S of integers, let $\operatorname{prod} S = \prod_{s \in S} s$ be the product of all elements of S. For instance, $\operatorname{prod} \{3, 6, 7\} = 3 \cdot 6 \cdot 7 = 126$.

Let $n \in \mathbb{N}$. Let $[n] := \{1, 2, \ldots, n\}$. Prove that

$$\sum_{L \text{ is a lacunar subset of } [n]} (\operatorname{prod} L)^2 = (n+1)!.$$

5.2 Solution

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6 Exercise 6

6.1 Problem

Let n be a positive integer, and let $a_1, a_2, \ldots, a_{n-1}$ be any $n-1$ real numbers.

Define an $(n+1)$-tuple $(u_0, u_1, \ldots, u_n)$ of real numbers recursively by $u_0 = 1$ and $u_1 = 1$ and

$$u_k = u_{k-1} + a_{k-1} u_{k-2} \qquad \text{for all } k \geq 2.$$

Define an $(n+1)$-tuple $(v_0, v_1, \ldots, v_n)$ of real numbers recursively by $v_0 = 1$ and $v_1 = 1$ and

$$v_k = v_{k-1} + a_{n-k+1} v_{k-2} \qquad \text{for all } k \geq 2.$$

Prove that $u_n = v_n$.

6.2 Hint

Compute $u_n = v_n$ for small n and see if you can guess a combinatorial formula.

6.3 Solution

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7 Exercise 7

7.1 Problem

Find formulas for $\sum_{k=0}^{n} \binom{n}{2k+1} 4^k$ and $\sum_{k=0}^{n} \binom{n}{2k} 4^k$ for each $n \in \mathbb{N}$.

7.2 Hint

This is a variation on Exercise 4.5.9 in the notes.

7.3 Solution

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8 Exercise 8

8.1 Problem

Let $(f_0, f_1, f_2, \ldots)$ be the Fibonacci sequence. Furthermore, set $f_{-1} := 1$ and $\ell_n := f_{n+1} + f_{n-1}$ for each $n \in \mathbb{N}$. Let $n, k \in \mathbb{N}$ be such that $n \geq k$. Show that

$$\ell_k f_n = (-1)^k f_{n-k} + f_{n+k}.$$

(Note that this generalizes Exercise 4.4.4 **(c)** in the notes, since $\ell_4 = 7$.)

8.2 Solution

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References