Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2024: Homework 2

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Please solve 4 of the 8 exercises!
Deadline: October 16, 2024

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1 Exercise 1

1.1 Problem

Let $(a_0, a_1, a_2, \ldots)$ be a sequence of positive integers obtained as follows: Choose $a_0 \in \{1, 2, 3, \ldots\}$ arbitrarily. For each $i > 0$, let a_i be the smallest positive integer that does not divide a_{i-1}. (For instance, if $a_0 = 12$, then $a_1 = 5$ and $a_2 = 2$.)

(a) Show that for each choice of a_0, there exists an $i > 0$ that satisfies $a_i = 2$.

(b) Let us denote the smallest such i as $f(a_0)$ (since it depends on a_0). What is the largest possible $f(a_0)$ as a_0 varies over all positive integers?

1.2 Solution

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2 Exercise 2

2.1 Problem

Let n be an integer such that $2 \nmid n$ and $3 \nmid n$ and $5 \nmid n$. Prove that $240 \mid n^4 - 1$.

(Do not check all possible remainders modulo 240 by brute force!)

2.2 Solution

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3 Exercise 3

3.1 Problem

Let a, b, c be three integers such that $b \perp c$ (that is, b is coprime to c). Prove that

$$\gcd(a, bc) = \gcd(a, b) \cdot \gcd(a, c) \qquad \text{and}$$
$$|a| \cdot \operatorname{lcm}(a, bc) = \operatorname{lcm}(a, b) \cdot \operatorname{lcm}(a, c).$$

3.2 Hint

The case when $a = 0$ is easy enough to be left to the reader, but it should be mentioned (unless your proof really needs no modifications in this case). Also, if you have obtained the second equality with an a instead of $|a|$, you must have done something slightly wrong.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let $(f_0, f_1, f_2, \ldots)$ be the Fibonacci sequence (which, as we recall, starts with $f_0 = 0$ and $f_1 = 1$ and continues by the recursive rule $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$). Furthermore, set $f_{-1} := 1$ and $\ell_n := f_{n+1} + f_{n-1}$ for each $n \in \mathbb{N}$.

(a) Prove that each $m \in \mathbb{N}$ satisfies

$$\gcd(f_{2m}, f_{2m+1} - 1) = \begin{cases} f_m, & \text{if } m \text{ is even;} \\ \ell_m, & \text{if } m \text{ is odd.} \end{cases}$$

(b) Prove that each odd integer $n \in \mathbb{N}$ satisfies

$$\gcd\left(f_n, f_{n+1} - 1\right) = \begin{cases} 2, & \text{if } 3 \mid n; \\ 1, & \text{if } 3 \nmid n. \end{cases}$$

4.2 Solution

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5 Exercise 5

5.1 Problem

Fix two positive integers p and q such that q is even. Let $(F_0, F_1, F_2, \ldots)$ be the sequence of integers defined by

$$F_n = p^{q^n} + 1 \qquad \text{for each } n \in \mathbb{N}.$$

(a) Prove that $\gcd\left(F_n, F_m\right) \in \{1, 2\}$ for any two distinct nonnegative integers n and m.

(b) Give a simple criterion for when this gcd is 1 and when it is 2.

5.2 Solution

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6 Exercise 6

6.1 Problem

Let a, b, c be three integers. Prove that $a^2 + b^2 + c^2 \not\equiv 7 \mod 8$.

6.2 Solution

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7 Exercise 7

7.1 Problem

The Midchopper is an obscure drawing tool that can mark the midpoint of any given segment.

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The Sidechopper is an even more obscure drawing tool that can subdivide any given segment in the ratio $n : (n+1)$ for any given choice of positive integer n (that is, you can choose two points A and B and a positive integer n, and the tool will mark the point C on the segment AB that satisfies $|AC| / |CB| = n / (n+1)$).

(a) Prove that using both of these tools together, you can subdivide any segment into n equal parts for any positive integer n.

(b) Prove that you cannot do this using the Midchopper alone, without the Sidechopper.

(c) Prove that you also cannot do this using the Sidechopper alone, without the Midchopper.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let n and m be two coprime positive integers. Let $u, v \in \mathbb{Z}$. Prove that

$$(u^n - v^n)(u^m - v^m) \mid (u - v)(u^{nm} - v^{nm}).$$

8.2 Hint

This generalizes Exercise 4.5.3 in the notes; its solution might be an inspiration.

8.3 Solution

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References