Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2024: Homework 1

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Please solve 4 of the 8 exercises!
Deadline: October 9, 2024

1 Exercise 1

1.1 Problem

Let $(f_0, f_1, f_2, \ldots)$ be the Fibonacci sequence (which, as we recall, starts with $f_0 = 0$ and $f_1 = 1$ and continues by the recursive rule $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$). Furthermore, set $f_{-1} := 1$ and $\ell_n := f_{n+1} + f_{n-1}$ for each $n \in \mathbb{N}$.

(a) Prove that $\ell_0 = 2$ and $\ell_1 = 1$ and $\ell_n = \ell_{n-1} + \ell_{n-2}$ for all $n \geq 2$.

Let $m \in \mathbb{N}$. Prove the following:

(b) We have $f_{2m} = f_m \ell_m$.

(c) We have $f_{2m+1} - 1 = f_m \ell_{m+1}$ if m is even.

(d) We have $f_{2m+1} - 1 = f_{m+1} \ell_m$ if m is odd.

1.2 Solution

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2 Exercise 2

2.1 Problem

Define a sequence $(a_0, a_1, a_2, \ldots)$ of positive rational numbers recursively by setting

$$a_0 = 1, \qquad a_1 = 1, \qquad a_2 = 1, \qquad \text{and}$$
$$a_n = \frac{a_{n-1}a_{n-2}}{a_{n-3}+1} \qquad \text{for each } n \geq 3.$$

Prove that a_n is the reciprocal of a positive integer for each $n \in \mathbb{N}$. (For instance, $a_8 = \frac{1}{195840}$. Note the subtle difference to Homework set #0 Exercise 3.)

2.2 Solution

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3 Exercise 3

3.1 Problem

For each $n \in \mathbb{N}$, let $[n]$ denote the set $\{1, 2, \ldots, n\}$.

If P is a finite set of integers and if $i \in [|P|]$, then $\min_i P$ shall denote the i-th smallest element of P. For instance, $\min_3 \{2, 4, 6, 8, 10, 12\} = 6$.

If S and T are two finite sets of integers, then we say that $S \preccurlyeq T$ if and only if we have

$$|S| \geq |T| \qquad \text{and} \qquad (\min_i S \leq \min_i T \text{ for all } i \in [|T|]) .$$

For instance, $\{3, 5, 8, 10\} \preccurlyeq \{4, 5, 9\}$.

Let S and T be two finite sets of positive integers. Prove the following:

(a) We have $S \preccurlyeq T$ if and only if each positive integer k satisfies $|S \cap [k]| \geq |T \cap [k]|$.

(b) Let p be a positive integer. Assume that $S \preccurlyeq T$. Show that $[p] \setminus T \preccurlyeq [p] \setminus S$.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $a_1, a_2, \ldots, a_n$ be n distinct positive integers. Prove that

$$a_1^2 + a_2^2 + \cdots + a_n^2 \geq \frac{2n+1}{3} \left(a_1 + a_2 + \cdots + a_n\right).$$

(Note that this is an equality when $a_i = i$ for all i, due to the formulas

$$1 + 2 + \cdots + n = \frac{n\left(n+1\right)}{2} \qquad \text{and} \qquad 1^2 + 2^2 + \cdots + n^2 = \frac{n\left(n+1\right)\left(2n+1\right)}{6}.$$

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4.2 Solution

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5 Exercise 5

5.1 Problem

Using Euler's famous formula $\sum\limits_{i=1}^{\infty} \frac{1}{i^2} = \frac{\pi^2}{6}$ (the answer to the Basel problem), prove that

$$\sum_{i=1}^{\infty} \frac{1}{i^2\left(i+1\right)} = \frac{\pi^2}{6} - 1 \qquad \text{and} \qquad \sum_{i=2}^{\infty} \frac{1}{i^2\left(i-1\right)} = 2 - \frac{\pi^2}{6}.$$

5.2 Solution

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6 Exercise 6

6.1 Problem

Let $n \geq 3$ be an integer. Prove that the number 1 can be written as

$$1 = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$$

for some n distinct positive integers $a_1, a_2, \ldots, a_n$. (For instance, for $n = 3$, we can write $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{6}$.)

6.2 Solution

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7 Exercise 7

7.1 Problem

Let n be a positive integer. Let $a_1, a_2, \ldots, a_n$ be n distinct positive reals. Consider the 2^n numbers of the form $\pm a_1 \pm a_2 \pm \cdots \pm a_n$. (There are n many $\pm$ signs here, and each one can be either a $+$ or a $-$ independently of the others; thus, there are 2^n possible choices.)

Prove that at least $\dbinom{n+1}{2}$ of these 2^n numbers are distinct.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let $(x_1, x_2, x_3, \ldots)$ be a sequence of integers that satisfies

$$\begin{aligned} x_1 &= 1 \qquad \text{and} \\ x_{2k} &= -x_k \qquad \text{for all } k \geq 1, \qquad \text{and} \\ x_{2k-1} &= (-1)^{k+1} x_k \qquad \text{for all } k \geq 1. \end{aligned}$$

(a) Prove that $x_1 + x_2 + \cdots + x_n = x_{n+1} + x_{n+2} + \cdots + x_{4n}$ for each $n \in \mathbb{N}$.

(b) Prove that $x_1 + x_2 + \cdots + x_n \geq 0$ for each $n \in \mathbb{N}$.

8.2 Solution

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References