Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2024: Homework 0

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Diagnostic homework set graded on completion. Please attempt to write meaningful responses (not necessarily complete solutions) to all exercises!
Deadline: September 25, 2024

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1 Exercise 1

1.1 Problem

Have you participated in mathematical contests, coding challenges, puzzle competitions and the likes? If so, which ones, and which were your favorites?

1.2 Solution

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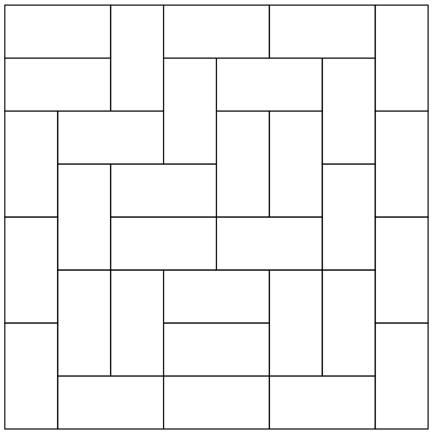
2 Exercise 2

2.1 Problem

A *domino* shall mean a 1×2-rectangle or a 2×1-rectangle. A board shaped like a 8×8-square can easily be tiled with dominoes, e.g., as follows:



("Tiling" a board by dominos means covering it by dominos in such a way that two dominos cannot have any overlap except along their edges. The dominos should lie with their sides parallel to the horizontal and vertical axes, and should be "snapped to the grid" – i.e., their edges should fall along the gridlines that subdivide the board into 1×1-squares.)

- **(a)** If we remove one of the four corner cells from our board, then can the remaining shape still be tiled with dominoes? (A "cell" means a 1×1-square.)
- **(b)** Consider two corner cells of the board that lie on the same side (e.g., the southwestern and the northwestern corners). If we remove these two cells from the board, then can the remaining shape still be tiled with dominoes?
- **(c)** Now consider two opposite corner cells of the board (e.g., the southwestern and the northeastern corners). If we remove these two cells from the board, then can the remaining shape still be tiled with dominoes?

2.2 Solution

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3 Exercise 3

3.1 Problem

Define a sequence $(a_0, a_1, a_2, \ldots)$ of positive rational numbers recursively by setting

$$a_0 = 1, \qquad a_1 = 1, \qquad a_2 = 1, \qquad \text{and}$$
$$a_n = \frac{a_{n-1}a_{n-3}}{a_{n-2}+1} \qquad \text{for each } n \geq 3.$$

Prove that a_n is the reciprocal of a positive integer for each $n \in \mathbb{N}$. (For instance, $a_8 = \frac{1}{448}$.)

3.2 Solution

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4 Exercise 4

4.1 Problem

Let n be a positive integer. Consider the $n-1$ fractions $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}, \ldots, \frac{n}{n-1}$ (that is, $\frac{k}{k-1}$ for all $k \in \{2, 3, \ldots, n\}$).

(a) Prove that the product of these $n-1$ fractions is n.

(b) For which n is it possible to invert some of these fractions so that the product of the $n-1$ fractions (the inverted and the non-inverted ones multiplied together) becomes 1 ?

(To "invert" a fraction means to swap its numerator with its denominator. For instance, for $n = 4$, we have the 3 fractions $\frac{2}{1}, \frac{3}{2}, \frac{4}{3}$. Inverting the first of them, we get the three fractions $\frac{1}{2}, \frac{3}{2}, \frac{4}{3}$, whose product is $\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{4}{3} = 1$, as desired.)

4.2 Solution

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5 Exercise 5

5.1 Problem

(a) Prove that $\sin 1^\circ + \sin 2^\circ + \cdots + \sin 179^\circ = \cot(0.5^\circ)$. (Recall that $\cot x := \frac{\cos x}{\sin x}$ for any angle x.)

(b) Prove that $1 \sin 1^\circ + 2 \sin 2^\circ + \cdots + 179 \sin 179^\circ = 90 \cot(0.5^\circ)$.

5.2 Solution

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6 Exercise 6

6.1 Problem

Prove that for each $n \in \mathbb{N}$, there exists a permutation $(p_1, p_2, \ldots, p_n)$ of the n-tuple $(1, 2, \ldots, n)$ with the property that each partial sum $p_1 + p_2 + \cdots + p_k$ (with $k \in \{1, 2, \ldots, n-1\}$) is divisible by p_{k+1}.

[**Example:** For $n = 7$, one such permutation is $(5, 1, 6, 4, 2, 3, 7)$.]

6.2 Solution

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7 Exercise 7

7.1 Problem

Let δ be a positive real number. Let $a_1, a_2, \ldots, a_n$ be n real numbers such that $n \geq 2$ and $a_1 + a_2 + \cdots + a_n = 0$ and

$$|a_i - a_{i+1}| \leq 2\delta \qquad \text{for all } i \in \{1, 2, \ldots, n\},$$

where we let a_{n+1} denote a_1. Prove that there exist at least two numbers $i \in \{1, 2, \ldots, n\}$ such that $|a_i| \leq \delta$.

7.2 Solution

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8 Exercise 8

8.1 Problem

(a) Let ABC be a triangle. Prove the inequality

$$\sin A + \sin B + \sin C \leq \frac{3\sqrt{3}}{2}.$$

Here, the A, B, C in the inequality denote the angles of triangle ABC (that is, $A = \measuredangle CAB$ and so on).

(b) Is there an analogous inequality for convex quadrilaterals? for convex n-gons?

8.2 Solution

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9 Exercise 9

9.1 Problem

Let n be a positive integer. Consider an $n \times n$-matrix A with real entries. An entry of A will be called *weird* if it is simultaneously (strictly) greater than the average of the entries in its column and (strictly) smaller than the average of the entries in its row. What is the largest possible number of weird entries that A can have?

[**Remark:** Of course, the weirdness of an entry depends not just on the entry itself but also on its position in A. Thus, when we speak of weird entries, we really mean weird cells.]

9.2 Solution

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10 Exercise 10

10.1 Problem

Let $n > 1$ be a integer. Let m be the smallest positive integer such that $1 + 2 + \cdots + m$ is divisible by n. (For example, if $n = 5$, then $m = 4$, since $1 + 2 + 3 + 4 = 10$ is divisible by 5 whereas 1 and $1 + 2$ and $1 + 2 + 3$ are not.)

Prove that $m > n$ if and only if n is a power of 2.

10.2 Solution

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References