Math 235: Mathematical Problem Solving: discussion session #3

Darij Grinberg

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Math 235 Fall 2024 discussion session #3

Topic: Farey series.

Define a **proper fraction** to be a rational number between 0 and 1 inclusive. By default, write every rational number as a reduced fraction, i.e., as $\frac{a}{b}$ with $b > 0$ and $a \perp b$.

For any positive integer n, let us list all proper fractions with denominator $\leq n$ in increasing order. Let F_n be this list; it is called the n**-th Farey series**. For example:

$$F_1 = \left(\frac{0}{1} < \frac{1}{1}\right);$$
$$F_2 = \left(\frac{0}{1} < \frac{1}{2} < \frac{1}{1}\right);$$
$$F_3 = \left(\frac{0}{1} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{1}{1}\right);$$
$$F_4 = \left(\frac{0}{1} < \frac{1}{4} < \frac{1}{3} < \frac{1}{2} < \frac{2}{3} < \frac{3}{4} < \frac{1}{1}\right);$$
$$F_5 = \left(\frac{0}{1} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{1}{1}\right);$$
$$F_6 = \left(\frac{0}{1} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \frac{2}{3} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \frac{1}{1}\right);$$
$$F_7 = \left(\frac{0}{1} < \frac{1}{7} < \frac{1}{6} < \frac{1}{5} < \frac{1}{4} < \frac{2}{7} < \frac{1}{3} < \frac{2}{5} < \frac{3}{7} < \frac{1}{2} < \frac{4}{7} \right.$$
$$\left. < \frac{3}{5} < \frac{2}{3} < \frac{5}{7} < \frac{3}{4} < \frac{4}{5} < \frac{5}{6} < \frac{6}{7} < \frac{1}{1}\right).$$

When constructing this sequence, you often have to cross-multiply fractions to compare them:

$$\left(\frac{a}{b} < \frac{c}{d}\right) \Longleftrightarrow (ad < bc)$$

(for $b, d > 0$). Making these tests, you may observe the following: Any two adjacent fractions $\frac{a}{b}, \frac{c}{d}$ in the above list satisfy $ad - bc = -1$. So we suspect the following:

> *Observation 1:* Let us say that two fractions $\frac{a}{b}, \frac{c}{d}$ are **good neighbors** if $ad - bc = -1$. Then, any two consecutive fractions in the list F_n are good neighbors.

Is this true? How can we prove it?

Something else: Consider F_n, and look at the fractions that are "new" in F_n (that is, have denominator n). For example, for $n = 5$, we have

$$F_5 = \left(\frac{0}{1} < \boxed{\frac{1}{5}} < \frac{1}{4} < \frac{1}{3} < \boxed{\frac{2}{5}} < \frac{1}{2} < \boxed{\frac{3}{5}} < \frac{2}{3} < \frac{3}{4} < \boxed{\frac{4}{5}} < \frac{1}{1}\right).$$

We observe that each "new" fraction is the **mediant** of its two neighbors.

The **mediant** of two fractions $\frac{a}{b}$ and $\frac{c}{d}$ (both in reduced form) is defined to be the fraction $\frac{a+c}{b+d}$. So we observe:

> *Observation 2:* Let $n > 1$. Then, any fraction with denominator n that appears in F_n is the mediant of its two neighbors in F_n. Moreover, these neighbors already appear in F_{n-1}.

Is this true? How do we prove this?

We also observe that F_n is symmetric around the middle: If x is the k-th fraction in F_n from the left, then $1 - x$ is the k-th fraction in F_n from the right. This is actually obvious. This is quite clear.

So let us prove our two observations. It is hard to argue about F_n given the way we have defined it – the definition is slick but not very usable. Let us instead work backwards: We redefine F_n through the observations, and then show that the list actually contains all the proper fractions with the denominator $\leq n$ in increasing order.

So we define, for every $n \in \mathbb{N}$, a list F'_n of proper fractions recursively:

- We let F'_1 be $\left(\frac{0}{1}, \frac{1}{1}\right)$.
- For any $n > 1$, if F'_{n-1} is defined, then we define F'_n as follows: For any two neighboring fractions $\frac{a}{b}$ and $\frac{c}{d}$ in F'_{n-1}, if $b + d = n$, then we insert the mediant $\frac{a+c}{b+d}$ between them.

We shall now show that the list F'_n

- is strictly increasing;
- consists of reduced fractions;
- contains each proper fraction with denominator $\leq n$.

If we can show all of this, then it will follow that this list F'_n is our Farey series F_n. This will yield Observation 2 immediately, and Observation 1 will also follow with a bit of work.

Let us get to work. Since our lists F'_n are defined recursively, we expect to prove everything by induction on n. We begin with the strict increase:

> *Claim 3:* The list F'_n is strictly increasing for each $n \geq 1$.

Proof. This is true for $n = 1$ by definition. For the induction step, it suffices to show that if $\frac{a}{b} < \frac{c}{d}$ (with $b, d > 0$), then $\frac{a}{b} < \frac{a+c}{b+d} < \frac{c}{d}$.

But this is very easy:

$$\begin{aligned}\left(\frac{a}{b} < \frac{a+c}{b+d}\right) &\Longleftarrow (a(b+d) < b(a+c)) \\ &\Longleftarrow (ab + ad < ab + bc) \\ &\Longleftarrow (ad < bc) \Longleftarrow \left(\frac{a}{b} < \frac{c}{d}\right).\end{aligned}$$

Similarly we can prove $\frac{a+c}{b+d} < \frac{c}{d}$. □

More or less the same argument proves a stronger claim:

> *Claim 4:* For any $n \geq 1$, any two consecutive fractions in F_n' are good neighbors.

Proof. This is true for $n = 1$ by definition. For the induction step, it suffices to show that if $\frac{a}{b}$ and $\frac{c}{d}$ are good neighbors, then so are $\frac{a}{b}$ and $\frac{a+c}{b+d}$, and so are $\frac{a+c}{b+d}$ and $\frac{c}{d}$. But this is very easy:

$$\begin{aligned}&\left(\frac{a}{b} \text{ and } \frac{a+c}{b+d} \text{ are good neighbors}\right) \\ \Longleftarrow\ &(a(b+d) - b(a+c) = -1) \\ \Longleftarrow\ &(ad - bc = -1) \\ \Longleftarrow\ &\left(\frac{a}{b} \text{ and } \frac{c}{d} \text{ are good neighbors}\right).\end{aligned}$$

Similarly we can handle $\frac{a+c}{b+d}$ and $\frac{c}{d}$. □

> *Claim 5:* For any $n \geq 1$, any fraction in F_n' is reduced.

Proof. Consider a fraction $\frac{a}{b}$ in F_n'. We must show that it is reduced. If it is $\frac{1}{1}$, this is clear. Otherwise, it is followed by another fraction $\frac{c}{d}$ in F_n'. By Claim 4, these two fractions $\frac{a}{b}$ and $\frac{c}{d}$ are good neighbors. That is, $ad - bc = -1$. If $g = \gcd(a, b)$, then g divides a and b and thus also divides $ad - bc = -1$, which entails that $g = 1$. Thus, $\gcd(a, b) = 1$. In other words, the fraction $\frac{a}{b}$ is reduced. □

We next need another property of mediants:

> *Claim 6:* Let $\frac{a}{b}$ and $\frac{c}{d}$ be two reduced fractions that are good neighbors. Let $\frac{x}{y}$ be another fraction (with x, y positive integers) such that $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$. Then, $y \geq b + d$.

In other words, among all the rational numbers that lie strictly between the two good neighbors $\frac{a}{b}$ and $\frac{c}{d}$, the mediant $\frac{a+c}{b+d}$ has the smallest denominator.

Proof of Claim 6. From $\frac{a}{b} < \frac{x}{y}$, we obtain $ay < bx$, so that $ay - bx < 0$. Since both sides are integers, this entails $ay - bx \leq -1$, or equivalently $bx - ay \geq 1$.

From $\frac{x}{y} < \frac{c}{d}$, we obtain $dx < cy$, so that $cy - dx > 0$. Hence, $cy - dx \geq 1$.

So we have found the two inequalities $bx - ay \geq 1$ and $cy - dx \geq 1$.

We eliminate x from these two inequalities by multiplying the former with d and the latter with b and adding them:

$$d \underbrace{(bx - ay)}_{\geq 1} + b \underbrace{(cy - dx)}_{\geq 1} \geq d + b = b + d.$$

Since

$$d (bx - ay) + b (cy - dx) = - \underbrace{(ad - bc)}_{\substack{=-1 \\ \text{(since } \frac{a}{b} \text{ and } \frac{c}{d} \\ \text{are good neighbors)}}} y = -(-1) y = y,$$

this rewrites as $y \geq b + d$, proving Claim 6. $\square$

> *Claim 7:* Let $\frac{a}{b}$ and $\frac{c}{d}$ be two reduced fractions that are good neighbors. Let $\frac{x}{y}$ be another fraction (with x, y positive integers) such that $\frac{a}{b} < \frac{x}{y} < \frac{c}{d}$. Assume that $y = b + d$. Then, $x = a + c$.

Proof. Assume the contrary. Thus, $x < a + c$ or $x > a + c$.

- In the former case, $x \leq a + c - 1$ and therefore $\frac{a}{b} < \frac{x}{y} \leq \frac{a + c - 1}{b + d}$ (since $y = b + d$). Thus, $a (b + d) < b (a + c - 1)$. Expanding this, we obtain $ab + ad < ba + bc - \underbrace{b}_{\geq 1} \leq ba + bc - 1$. Upon cancellation of ab, this becomes $ad < bc - 1$. But this contradicts $ad - bc = -1$ (which is because $\frac{a}{b}$ and $\frac{c}{d}$ are good neighbor).

- In the latter case, $x \geq a+c+1$ and therefore $\frac{c}{d} > \frac{x}{y} \geq \frac{a+c+1}{b+d}$. Thus, $c(b+d) > (a+c+1)d$. Upon expanding and cancelling cd, we transform this into $cb > ad+d \geq ad+1$, thus $ad < bc-1$, which again contradicts the good neighborliness of $\frac{a}{b}$ and $\frac{c}{d}$.

So we found a contradiction in both cases. $\square$

Now the hard part:

Claim 8: For any $n \geq 1$, the list F'_n contains each reduced proper fraction with denominator $\leq n$.

Proof. Consider a reduced proper fraction $\frac{p}{q}$ with denominator $q \leq n$. We want to show that it appears in F'_n. It suffices to show that it apears in F'_q, since F'_n is just F'_q with some extra fractions inserted.

We shall prove this by strong induction on q.

Induction step: Assume that each reduced proper fraction $\frac{u}{v}$ with denominator $< q$ already appears in the corresponding F'_v. We must prove that $\frac{p}{q}$ appears in F'_q.

WLOG assume that $q > 1$, since otherwise $\frac{p}{q}$ is $\frac{0}{1}$ or $\frac{1}{1}$ and already appears in F'_1.

We expect $\frac{p}{q}$ to appear as the mediant $\frac{a+c}{b+d}$ for some good neighbors $\frac{a}{b}$ and $\frac{c}{d}$. What should these $\frac{a}{b}$ and $\frac{c}{d}$ be?

Clearly, the denominators b and d should be $\leq q-1$ (since they are positive and their sum $b+d$ should be q). So the fractions $\frac{a}{b}$ and $\frac{c}{d}$ should appear in F'_{q-1}. Moreover, these fractions should be consecutive in F'_{q-1}, so that their mediant $\frac{a+c}{b+d}$ gets inserted in F'_q.

This gives us an idea how to find $\frac{a}{b}$ and $\frac{c}{d}$: We let $\frac{a}{b}$ be the largest fraction in F'_{q-1} that is $< \frac{p}{q}$, and we let $\frac{c}{d}$ be the smallest fraction in F'_{q-1} that is $> \frac{p}{q}$. These two fractions $\frac{a}{b}$ and $\frac{c}{d}$ must be consecutive in F'_{q-1}, since $\frac{p}{q}$ does not appear in F'_{q-1} (its denominator is too high). Therefore, they are good neighbors (by Claim 4), and we have $\frac{a}{b} < \frac{p}{q} < \frac{c}{d}$. Thus, Claim 6 yields $q \geq b+d$.

If $b+d$ was $< q$, then b and d would be $\leq b+d-1$ (since b and d are positive), and therefore would already appear in F'_{b+d-1} (by the induction hypothesis), which would cause their mediant $\frac{a+c}{b+d}$ to already be inserted in F'_{b+d} and thus appear in

F'_{q-1} (since $b+d \leq q-1$), which would contradict the fact that they are consecutive in F'_{q-1}. So $q \geq b+d$ but $b+d$ is not $< q$. Therefore, $q = b+d$. Therefore, Claim 7 yields $p = a+c$. Hence, $\frac{p}{q} = \frac{a+c}{b+d}$, which is the mediant of the consecutive fractions $\frac{a}{b}$ and $\frac{c}{d}$ in F'_{q-1} and thus gets inserted in F'_q. So we have shown that $\frac{p}{q}$ appears in F'_q. This completes the induction, and proves Claim 8. $\square$

Claims 3 and 8 together entail that F'_n is the list of all reduced fractions between 0 and 1 with denominator $\leq n$ in increasing order. In other words, $F'_n = F_n$. Thus, Observation 1 and Observation 2 follow.

Here come three more properties of the Farey series:

Exercise 0.0.1. For any $n \geq 1$, show that

$$\sum_{\substack{\frac{a}{b} \text{ and } \frac{c}{d} \text{ are two} \\ \text{consecutive fractions in } F_n}} \frac{1}{bd} = 1.$$

Exercise 0.0.2. Prove the following claim: If $\frac{a}{b}$ and $\frac{c}{d}$ are two reduced fractions in the interval $[0,1]$ that are good neighbors (i.e., that satisfy $ad - bc = -1$), then $\frac{a}{b}$ and $\frac{c}{d}$ appear as consecutive entries of the Farey series F'_n for some $n \in \mathbb{N}$ (indeed, for all n with $\max\{b,d\} \leq n < b+d$).

Exercise 0.0.3. Prove that **any** fraction in F_n, except for the leftmost and the rightmost, is the mediant of its two neighbors (upon reducing).

(For example, in $F_5 = \left(\cdots < \frac{2}{5} < \frac{1}{2} < \frac{3}{5} < \cdots \right)$, the fraction $\frac{1}{2}$ is the mediant of its two neighbors $\frac{2}{5}$ and $\frac{3}{5}$, since the latter mediant is $\frac{2+3}{5+5} = \frac{5}{10} = \frac{1}{2}$.)

Various other properties of the Farey series can be found, e.g., in [GrKnPa94, §4.5 and §4.9]. For example, the length (= number of entries) of F_n (that is, the number of all reduced fractions in $[0,1]$ with denominator $\leq n$) is approximately $\frac{3n^2}{\pi^2}$ (up to an error of $O(n \log n)$).

A more geometric view of Farey series can be found in [Hatche24, Chapter 1].

References

[GrKnPa94] Ronald L. Graham, Donald E. Knuth, Oren Patashnik, *Concrete Mathematics, Second Edition*, Addison-Wesley 1994.

See https://www-cs-faculty.stanford.edu/~knuth/gkp.html for errata.

[Hatche24] Allen Hatcher, *Topology of Numbers*, AMS, September 2024.
https://pi.math.cornell.edu/~hatcher/TN/TNbook.pdf