

Math 221 Winter 2023 (Darij Grinberg): midterm 2

due date: Sunday 2023-03-05 at 11:59PM on gradescope (
<https://www.gradescope.com/courses/487830>).

Please solve only **4 of the 6 exercises**.

NO collaboration allowed – this is a midterm!

(But you can still ask me questions.)

Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$.

Theorem 3.6.3 in Lecture 10 shows that if p is a prime, then all the binomial coefficients $\binom{p}{i}$ in the p -th row of Pascal's triangle are divisible by p (except for the two 1's on the borders of the triangle). The following exercise, in contrast, claims that the binomial coefficients in the $(p-1)$ -st row are alternatingly congruent to 1 and to -1 modulo p :

Exercise 1. Let p be a prime. Prove that

$$\binom{p-1}{i} \equiv (-1)^i \pmod{p} \quad \text{for each } i \in \{0, 1, \dots, p-1\}.$$

[Hint: What connects the three binomial coefficients $\binom{p-1}{i}$, $\binom{p-1}{i-1}$ and $\binom{p}{i}$?]

Here is another divisibility related to prime numbers:

Exercise 2. Let p be a prime larger than 3. Prove that $p^2 \equiv 1 \pmod{24}$.

[Hint: Recall some older problems. Also note that the integers 3 and 8 are coprime.]

Now to some counting problems. The symbol “#” means “number”.

To “compute” a number means to find a closed-form expression for this number (with no summation signs) and to prove this formula. I expect proofs to be given at the level of detail and rigor seen in Lectures 12 and 13.

Exercise 3. Let $n \geq 2$ be an integer.

(a) Compute the # of subsets of $\{1, 2, \dots, n\}$ that contain both 1 and 2.

(b) Compute the # of 3-element subsets of $\{1, 2, \dots, n\}$ that contain both 1 and 2.

Exercise 4. Let $n \in \mathbb{N}$. Compute the # of all pairs $(a, b) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ satisfying $a \equiv b \pmod{2}$.

(The answer will depend on whether n is even or odd. You can find a unified formula using the floor of a number, but you don't have to.)

Exercise 5. Let $n \in \mathbb{N}$. Compute the # of all pairs $(a, b) \in \{1, 2, \dots, n\} \times \{1, 2, \dots, n\}$ satisfying $2a \leq b$.

(Again, the answer will depend on whether n is even or odd.)

Finally, an exercise combining functions with number theory:

Exercise 6. For any positive integer d , let us define the function

$$\begin{aligned} r_d : \mathbb{Z} &\rightarrow \mathbb{Z}, \\ n &\mapsto n \% d \end{aligned}$$

(which sends each integer n to the remainder of the division of n by d). For example, $r_5(18) = 18 \% 5 = 3$ and $r_6(18) = 18 \% 6 = 0$.

(a) Make a table of the values of the function $r_2 \circ r_3$ on the inputs $0, 1, 2, 3, 4, 5$.

(b) Prove that $r_2 \circ r_3 \neq r_2$.

(c) Let d and e be two positive integers such that $d \mid e$. Prove that $r_d \circ r_e = r_d$.
