Math 221 Winter 2023 (Darij Grinberg): midterm 1
due date: Sunday 2023-02-14 at 11:59PM on gradescope (
`https://www.gradescope.com/courses/487830`).
Please solve only **4 of the 6 exercises**.
NO collaboration allowed – this is a midterm!
(But you can still ask me questions.)

Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Exercise 1. Let $n \in \mathbb{N}$, and let q be any number distinct from 1. Prove that

$$\sum_{k=1}^{n} kq^k = q \cdot \frac{nq^{n+1} - (n+1)\, q^n + 1}{(q-1)^2}.$$

Exercise 2. Let $(a_0, a_1, a_2, \ldots)$ be a sequence of integers defined recursively by

$$a_0 = 2, \qquad a_1 = 1,$$
$$a_n = a_{n-1} + 6a_{n-2} \qquad \text{for all } n \geq 2.$$

Prove that $a_n = 3^n + (-2)^n$ for each $n \in \mathbb{N}$.

Exercise 3. Let $n \in \mathbb{N}$. Prove that

$$\underbrace{1 + 2 + \cdots + n}_{= \sum_{k=1}^{n} k} = \underbrace{n^2 - (n-1)^2 + (n-2)^2 - (n-3)^2 \pm \cdots + (-1)^{n-1} 1^2}_{= \sum_{k=1}^{n} (-1)^{n-k} k^2}.$$

Exercise 4. Let $n \in \mathbb{N}$. Prove that

$$\prod_{i=0}^{n} \binom{2i}{i} = 2^n \prod_{i=0}^{n} \binom{n+i}{n-i}.$$

Exercise 5. Let $(a_0, a_1, a_2, \ldots)$ be a sequence of integers defined recursively by

$$a_n = 1 + a_0 a_1 \cdots a_{n-1} \qquad \text{for all } n \geq 0.$$

(In particular, $a_0 = 1 + \underbrace{a_0 a_1 \cdots a_{0-1}}_{=(\text{empty product})=1} = 1 + 1 = 2$.) Here are the first few entries of this sequence:

n	0	1	2	3	4	5	6
a_n	2	3	7	43	1807	3263443	10650056950807

(notice the astronomical growth!).

(a) Prove that

$$a_{n+1} = a_n^2 - a_n + 1 \qquad \text{for each } n \geq 0.$$

(b) Prove that

$$\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_{n-1}} = 1 - \frac{1}{a_n - 1} \qquad \text{for each } n \geq 0.$$

Now, recall the Tower of Hanoi puzzle (as discussed in Lecture 1), and let m_n denote the # of moves needed to win (= solve this puzzle) with n disks. As we have seen in Theorem 1.2.2 (Lecture 2), we have $m_n = 2^n - 1$ for each $n \in \mathbb{N}$.

Consider the variant of the Tower of Hanoi puzzle in which we have 4 instead of 3 pegs, but otherwise the rules of the game are the same (and the goal is still is to move all n disks from peg 1 to peg 3). Let t_n denote the # of moves needed to win this variant with n disks.

Exercise 6. **(a)** Prove that

$$t_{a+b} \leq m_b + 2t_a \qquad \text{for any } a, b \in \mathbb{N}.$$

(b) Prove that $t_4 \leq 9$.