Math 221 Winter 2023, Lecture 16: Enumeration

website: `https://www.cip.ifi.lmu.de/~grinberg/t/23wd`

4. An introduction to enumeration

4.6. Counting, formally (cont'd)

4.6.2. Rules for sizes of finite sets

We have defined the size $|S|$ of a finite set S in §4.6.1 (Lecture 15). Let us now state some rules for these sizes that make them easier to compute. We will not prove these rules, as they are all obvious from common sense and their rigorous proofs would reasonably belong into a text on formalized foundations of mathematics.

The most important rule is the following:

Theorem 4.6.1 (Bijection Principle)**.** Let A and B be two finite sets. Then, $|A| = |B|$ if and only if there exists a bijection from A to B.

(As we recall, a "bijection" means a bijective map, and this is the same as a map that has an inverse.)

The next rule is obvious from one of our definitions of size:

Theorem 4.6.2. For each $n \in \mathbb{N}$, we have $|[n]| = n$.

Here, we recall that $[n]$ means the set $\{1, 2, \ldots, n\}$ consisting of the first n positive integers.

The next rule classifies sets of small size:

Theorem 4.6.3. Let S be a set. Then:

(a) We have $|S| = 0$ if and only if $S = \varnothing$ (that is, if S is empty).

(b) We have $|S| = 1$ if and only if $S = \{s\}$ for a single element s.

(c) We have $|S| = 2$ if and only if $S = \{s, t\}$ for two distinct elements s and t.

The next rule says that inserting a new element into a finite set increases the size of this set by 1:

Theorem 4.6.4. Let S be a finite set. Let t be an object such that $t \notin S$ (that is, t does not belong to S). Then,

$$|S \cup \{t\}| = |S| + 1.$$

Here are some more substantial facts:

Theorem 4.6.5 (Sum rule for two sets)**.** Let A and B be two disjoint finite sets. (Recall that "disjoint" means $A \cap B = \varnothing$.) Then, the set $A \cup B$ is again finite, and has size

$$|A \cup B| = |A| + |B| .$$

Theorem 4.6.6 (Sum rule for k sets)**.** Let $A_1, A_2, \ldots, A_k$ be k disjoint finite sets. (Recall that "disjoint" for k sets means that any two of them are disjoint, i.e., that $A_i \cap A_j = \varnothing$ for any $i < j$.) Then, the set $A_1 \cup A_2 \cup \cdots \cup A_k$ is finite, and has size

$$|A_1 \cup A_2 \cup \cdots \cup A_k| = |A_1| + |A_2| + \cdots + |A_k| .$$

Theorem 4.6.7 (Difference rule)**.** Let T be a subset of a finite set S. Then:

(a) The set T is finite, and its size $|T|$ satisfies $|T| \leq |S|$.

(b) We have $|S \setminus T| = |S| - |T|$.

(c) If $|T| = |S|$, then $T = S$.

The following theorem has been previously stated (without using the "size" terminology) as Theorem 4.4.8 in Lecture 13:

Theorem 4.6.8 (Product rule for two sets)**.** Let A and B be any finite sets. Then, the set

$$A \times B = \{\text{all pairs } (a, b) \text{ with } a \in A \text{ and } b \in B\}$$

is again finite and has size

$$|A \times B| = |A| \cdot |B| .$$

Likewise, the following theorem was Theorem 4.4.9 in Lecture 13:

Theorem 4.6.9 (Product rule for k sets)**.** Let $A_1, A_2, \ldots, A_k$ be any k finite sets. Then, the set

$$A_1 \times A_2 \times \cdots \times A_k = \{\text{all } k\text{-tuples } (a_1, a_2, \ldots, a_k) \text{ with } a_i \in A_i \text{ for each } i \in [k]\}$$

is again finite and has size

$$|A_1 \times A_2 \times \cdots \times A_k| = |A_1| \cdot |A_2| \cdot \cdots \cdot |A_k| .$$

All the above theorems are foundational, and are perhaps the reason why the arithmetic operations $+$, $-$ and $\cdot$ on nonnegative integers have been introduced some millennia ago. Nevertheless, they can be rigorously proved, but this is not something we will do here.[1]

The above theorems are known as "basic counting rules" or "counting principles". There are a few more counting principles, which we might state later on.

4.6.3. $A \cup B$ and $A \cap B$ revisited

As a first application of these rules, let us derive the following "generalized sum rule for two sets":

Theorem 4.6.10. Let A and B be two finite sets (not necessarily disjoint). Then,

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

Proof. We first claim that

$$(A \cup B) \setminus A = B \setminus (A \cap B). \tag{1}$$

This equality is obvious using Venn diagrams, but let us prove it rigorously using "element chasing":[2]

Proof of (1). Let us first prove $(A \cup B) \setminus A \subseteq B \setminus (A \cap B)$. In order to do so, we must show that each $x \in (A \cup B) \setminus A$ belongs to $B \setminus (A \cap B)$. Let us do this: Let $x \in (A \cup B) \setminus A$. Thus, $x \in A \cup B$ but $x \notin A$. From $x \in A \cup B$, we see that $x \in A$ or $x \in B$. But the first of these two possibilities is impossible (since $x \notin A$). Thus, the second possibility must hold. In other words, we have $x \in B$. Furthermore, we have $x \notin A \cap B$ (since $x \in A \cap B$ would entail $x \in A \cap B \subseteq A$, which would contradict $x \notin A$). Combining $x \in B$ with $x \notin A \cap B$, we obtain $x \in B \setminus (A \cap B)$.

Forget that we fixed x. We thus have shown that each $x \in (A \cup B) \setminus A$ belongs to $B \setminus (A \cap B)$. In other words, $(A \cup B) \setminus A \subseteq B \setminus (A \cap B)$.

Next, let us prove that $B \setminus (A \cap B) \subseteq (A \cup B) \setminus A$. To do so, we must show that each $x \in B \setminus (A \cap B)$ belongs to $(A \cup B) \setminus A$. We do this as follows: Let $x \in B \setminus (A \cap B)$. Thus, $x \in B$ but $x \notin A \cap B$. If we had $x \in A$, then we would have $x \in A \cap B$ (since $x \in A$ and $x \in B$), which would contradict $x \notin A \cap B$. Hence, we cannot have $x \in A$. Thus, $x \notin A$. Also, $x \in B \subseteq A \cup B$. Combining this with $x \notin A$, we find $x \in (A \cup B) \setminus A$.

Forget that we fixed x. We thus have shown that each $x \in B \setminus (A \cap B)$ belongs to $(A \cup B) \setminus A$. In other words, $B \setminus (A \cap B) \subseteq (A \cup B) \setminus A$.

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[1] For instance, Theorem 4.6.8 can be proved by induction on $|B|$ using Theorem 4.6.6 and Theorem 4.6.1, whereas Theorem 4.6.9 can be proved by induction on k using Theorem 4.6.8.

[2] It is worth noting that both sides of (1) are equal to $B \setminus A$. However, we will not need this fact.

Now, combining the two inclusions

$$(A \cup B) \setminus A \subseteq B \setminus (A \cap B) \qquad \text{and} \qquad B \setminus (A \cap B) \subseteq (A \cup B) \setminus A,$$

we obtain $(A \cup B) \setminus A = B \setminus (A \cap B)$. Thus, (1) is proved.] □

We now step to the counting. Taking sizes on both sides of (1), we obtain

$$|(A \cup B) \setminus A| = |B \setminus (A \cap B)|. \tag{2}$$

But A is a subset of $A \cup B$. Thus, the difference rule (Theorem 4.6.7 **(b)**, applied to $S = A \cup B$ and $T = A$) yields

$$|(A \cup B) \setminus A| = |A \cup B| - |A|. \tag{3}$$

Also, $A \cap B$ is a subset of B. Thus, the difference rule (Theorem 4.6.7 **(b)**, applied to $S = B$ and $T = A \cap B$) yields

$$|B \setminus (A \cap B)| = |B| - |A \cap B|. \tag{4}$$

But we know from (2) that the left hand sides of the two equalities (3) and (4) are equal. Thus, their right hand sides are also equal. In other words,

$$|A \cup B| - |A| = |B| - |A \cap B|.$$

Solving this for $|A \cup B|$, we find

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

This proves Theorem 4.6.10.

(A nicer proof can be given using finite sums; this is done, e.g., in [Math222, Lecture 19, §2.7].) □

Note that Theorem 4.6.10 has an analogue for three sets: If A, B, C are three finite sets, then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|.$$

More generally, such a formula can be stated for any k finite sets, and is known as the "principle of inclusion and exclusion" or "Sylvester's sieve formula". See [Math222, Lecture 19, §2.7] or any textbook on combinatorics.

4.7. Redoing some proofs rigorously

Previously (in Lectures 12 and 13), we have proved some results using informal counting arguments. Let us now revisit these results and make these arguments rigorous.

4.7.1. Integers in an interval

We recall that the notation $[a, b]$ means the set

$$\{a,\ a+1,\ a+2,\ \ldots,\ b\} = \{x \in \mathbb{Z} \ \mid \ a \leq x \leq b\}$$

whenever a and b are two integers. In particular, $[n] = [1, n]$ for every $n \in \mathbb{N}$.

We begin with Proposition 4.2.2 in Lecture 12 (rewritten using the notation $[a, b]$):

Proposition 4.7.1. Let $a, b \in \mathbb{Z}$ be such that $a \leq b + 1$.

Then, there are exactly $b - a + 1$ numbers in the set $[a, b]$. In other words, there are exactly $b - a + 1$ integers between a and b (inclusive).

In Lecture 12, we proved this informally by inducting on b. This proof can be trivially made rigorous; the induction step relies on Theorem 4.6.4 (because $[a, b+1] = [a, b] \cup \{b+1\}$ and $b + 1 \notin [a, b]$).

But there is also a more direct proof:

Second proof of Proposition 4.7.1. Consider the map

$$f : \underbrace{[b-a+1]}_{=\{1,2,\ldots,b-a+1\}} \to \underbrace{[a,b]}_{=\{a,a+1,\ldots,b\}}, \qquad i \mapsto i + (a-1).$$

This map f just adds $a - 1$ to its input. (Informally, we can view it as moving numbers to the right by $a - 1$ units on the number line.)

It is easy to see that this map f has an inverse: Namely, the map

$$[a, b] \to [b-a+1], \qquad j \mapsto j - (a-1)$$

is an inverse of f (since subtraction undoes addition). Thus, the map f is bijective (by Theorem 4.5.7 in Lecture 15), i.e., is a bijection. Hence, there is a bijection from $[b - a + 1]$ to $[a, b]$ (namely, f). The bijection principle (Theorem 4.6.1, applied to $A = [b - a + 1]$ and $B = [a, b]$) thus yields

$$|[b-a+1]| = |[a,b]|.$$

Hence,

$$|[a,b]| = |[b-a+1]| = b-a+1$$

(by Theorem 4.6.2, since $a \leq b + 1$ yields $b - a + 1 \in \mathbb{N}$). In other words, there are exactly $b - a + 1$ numbers in the set $[a, b]$. This proves Proposition 4.7.1 again. □

We could also reprove Proposition 4.2.1 in Lecture 12 rigorously, but (again) the proof we gave was already rigorous enough; we just need to rewrite it using Theorem 4.6.4.

4.7.2. Counting all subsets

Now, we recall Theorem 4.3.1 in Lecture 12 (but shorten it using the notation $[n]$ for $\{1, 2, \ldots, n\}$):

Theorem 4.7.2. Let $n \in \mathbb{N}$. Then,

$$(\text{\# of subsets of } [n]) = 2^n.$$

The proof we gave in Lecture 12 had some informal steps; let us now make it rigorous:[3]

Rigorous proof of Theorem 4.7.2. We induct on n.

The *base case* ($n = 0$) is easy: The set $[0]$ is empty, and thus its only subset is $\{\}$ itself; hence, the # of subsets of $[0]$ is $1 = 2^0$. In other words, Theorem 4.7.2 holds for $n = 0$.

Induction step: We proceed from $n - 1$ to n. Thus, let n be a positive integer. We assume (as the induction hypothesis) that Theorem 4.7.2 holds for $n - 1$ instead of n, and we set out to prove that it holds for n.

So our induction hypothesis says that

$$(\text{\# of subsets of } [n-1]) = 2^{n-1}.$$

Our goal is to prove that

$$(\text{\# of subsets of } [n]) = 2^n.$$

We define

- a **red set** to be a subset of $[n]$ that contains n;
- a **green set** to be a subset of $[n]$ that does not contain n.

For example, if $n = 3$, then the red sets are

$$\{3\}, \qquad \{1,3\}, \qquad \{2,3\}, \qquad \{1,2,3\},$$

whereas the green sets are

$$\{\}, \qquad \{1\}, \qquad \{2\}, \qquad \{1,2\}.$$

A set cannot be red and green at the same time. In other words, the sets

$$\{\text{red sets}\} \qquad \text{and} \qquad \{\text{green sets}\}$$

[3] Most of the proof below is copied verbatim from Lecture 12.

are disjoint[4]. Hence, the sum rule for two sets (Theorem 4.6.5, applied to $A = \{\text{red sets}\}$ and $B = \{\text{green sets}\}$) yields

$$|\{\text{red sets}\} \cup \{\text{green sets}\}| = |\{\text{red sets}\}| + |\{\text{green sets}\}|.$$

(This is just a formal way to say "the # of all sets that are red or green equals the # of red sets plus the # of green sets". Indeed, the notation $\{\text{red sets}\}$ means the **set** of all red sets, and thus the expression $|\{\text{red sets}\}|$ means the **size** of the set of all red sets, i.e., the # of all red sets.)

Furthermore, each subset of $[n]$ is either red or green (and conversely, each red or green set is a subset of $[n]$). Hence,

$$\{\text{subsets of } [n]\} = \{\text{red sets}\} \cup \{\text{green sets}\}.$$

Therefore,

$$\begin{aligned} |\{\text{subsets of } [n]\}| &= |\{\text{red sets}\} \cup \{\text{green sets}\}| \\ &= |\{\text{red sets}\}| + |\{\text{green sets}\}| \end{aligned}$$

(as we have proved above). In other words,

$$(\#\text{ of subsets of } [n]) = (\#\text{ of red sets}) + (\#\text{ of green sets}). \tag{5}$$

Thus it remains to count the red sets and the green sets separately.

The green sets are easy: They are just the subsets of $[n-1]$. Hence,

$$(\#\text{ of green sets}) = (\#\text{ of subsets of } [n-1]) = 2^{n-1}$$

(by the induction hypothesis).

Counting the red sets is trickier. In Lecture 12, we did this by setting up a one-to-one correspondence between the red sets and the green sets. Formally, a one-to-one correspondence is just a bijection. Thus, our one-to-one correspondence should become a bijection from $\{\text{green sets}\}$ to $\{\text{red sets}\}$ (i.e., from the set of all green sets to the set of all red sets).

As we recall, we obtained this correspondence as follows: To turn a green set red, we insert n into it; conversely, to turn a red set green, we remove n from it. Rigorously, this means that we define two maps

$$\begin{aligned} \text{ins}_n : \{\text{green sets}\} &\to \{\text{red sets}\}, \\ G &\mapsto G \cup \{n\} \end{aligned}$$

[4]Keep in mind: The notation "$\{\text{red sets}\}$" stands for **the set of** all red sets. For example, if $n = 3$, then

$$\{\text{red sets}\} = \{\{3\}, \{1,3\}, \{2,3\}, \{1,2,3\}\}.$$

and

$$\text{rem}_n : \{\text{red sets}\} \to \{\text{green sets}\}, \\ R \mapsto R \setminus \{n\}.$$

It is easy to see that both of these maps ins_n and rem_n are well-defined[5]. A little bit of set-theoretic computation shows that

$$\text{ins}_n(\text{rem}_n(R)) = R \qquad \text{for every red set } R$$

(because if R is a red set, then

$$\begin{aligned} \text{ins}_n(\text{rem}_n(R)) &= \underbrace{\text{rem}_n(R)}_{\substack{=R\setminus\{n\} \\ \text{(by the definition of } \text{rem}_n)}} \cup \{n\} \qquad \text{(by the definition of } \text{ins}_n) \\ &= (R \setminus \{n\}) \cup \{n\} = R \qquad \text{(since } n \in R \text{ (because } R \text{ is red))} \end{aligned}$$

). Similarly,

$$\text{rem}_n(\text{ins}_n(G)) = G \qquad \text{for every green set } G.$$

These two equalities show that the map rem_n is an inverse of ins_n. Hence, the map ins_n has an inverse, i.e., is bijective (by Theorem 4.5.7 in Lecture 15). In other words, ins_n is a bijection. Hence, there exists a bijection from $\{\text{green sets}\}$ to $\{\text{red sets}\}$ (namely, ins_n). Thus, the bijection principle yields

$$|\{\text{green sets}\}| = |\{\text{red sets}\}|.$$

In other words,

$$(\text{\# of green sets}) = (\text{\# of red sets}),$$

and thus

$$(\text{\# of red sets}) = (\text{\# of green sets}) = 2^{n-1}.$$

Combining what we have shown, we now obtain

$$\begin{aligned} (\text{\# of subsets of } [n]) &= \underbrace{(\text{\# of red sets})}_{=2^{n-1}} + \underbrace{(\text{\# of green sets})}_{=2^{n-1}} \\ &= 2^{n-1} + 2^{n-1} = 2 \cdot 2^{n-1} = 2^n. \end{aligned}$$

This is precisely what we needed to prove. This completes the induction step, and thus Theorem 4.7.2 is proved. □

[5] Indeed, we need to show that

- if G is a green set, then $G \cup \{n\}$ is a red set;
- if R is a red set, then $R \setminus \{n\}$ is a green set.

Both of these claims are very easy. For instance, if G is a green set, then G is a subset of $[n]$, and thus $G \cup \{n\}$ is a subset of $[n]$ as well (since $n \in [n]$), and furthermore is red (since $n \in \{n\} \subseteq G \cup \{n\}$).

Theorem 4.3.2 in Lecture 13 says:

Theorem 4.7.3. Let $n \in \mathbb{N}$. Let S be an n-element set. Then,

$$(\text{\# of subsets of } S) = 2^n.$$

Rigorous proof. Informally, we derived this from Theorem 4.7.2 by renaming the elements of S as $1, 2, \ldots, n$ (so that S became the set $[n]$).

Rigorously, this means setting up a one-to-one correspondence between the subsets of S and the subsets of $[n]$, and then using the bijection principle to argue that the # of the former equals the # of the latter.

How do we get this correspondence? First, we set up a one-to-one correspondence between the **elements** of S and the elements of $[n]$. (This is what the "renaming" in our informal proof was secretly doing.) Formally, this can be done as follows:

The set S is an n-element set, i.e., has size n. Hence, by the definition of size, the set S is isomorphic to $[n]$. In other words, there is a bijection $\alpha : S \to [n]$. Consider this α. Being a bijection, the map α has an inverse α^{-1} (by Theorem 4.5.7 in Lecture 15).

Now, define a map

$$\begin{aligned} \alpha_* : \{\text{subsets of } S\} &\to \{\text{subsets of } [n]\}, \\ T &\mapsto \{\alpha(t) \mid t \in T\}. \end{aligned}$$

Explicitly, this map α_* sends every subset $\{s_1, s_2, \ldots, s_k\}$ of S to the subset $\{\alpha(s_1), \alpha(s_2), \ldots, \alpha(s_k)\}$ of $[n]$; that is, it applies α to every element of the input subset. (For example, if $n = 3$ and $S = \{\text{"cat", "dog", "rat"}\}$ and if $\alpha(\text{"cat"}) = 1$ and $\alpha(\text{"dog"}) = 2$ and $\alpha(\text{"rat"}) = 3$, then $\alpha_*(\{\text{"cat", "rat"}\}) = \{1, 3\}$.)

Conversely, we can define a map

$$\begin{aligned} \left(\alpha^{-1}\right)_* : \{\text{subsets of } [n]\} &\to \{\text{subsets of } S\}, \\ T &\mapsto \left\{\alpha^{-1}(t) \mid t \in T\right\}. \end{aligned}$$

(This map $\left(\alpha^{-1}\right)_*$ is defined in the same way as α_*, but using the map α^{-1} instead of α. For example, if $n = 3$ and $S = \{\text{"cat", "dog", "rat"}\}$ and if $\alpha(\text{"cat"}) = 1$ and $\alpha(\text{"dog"}) = 2$ and $\alpha(\text{"rat"}) = 3$, then $\left(\alpha^{-1}\right)_*(\{2, 3\}) = \{\text{"dog", "rat"}\}$.)

It is easy to see that the map $\left(\alpha^{-1}\right)_*$ is an inverse of α_* (because applying α to each element of a given set and then applying α^{-1} to the results will recover the original set, and likewise if you apply α^{-1} first and then α). Thus, the map α_* has an inverse, i.e., is a bijection (by Theorem 4.5.7 in Lecture 15). Thus, we have found a bijection from $\{\text{subsets of } S\}$ to $\{\text{subsets of } [n]\}$ (namely, α_*). Hence, the bijection principle (Theorem 4.6.1) yields

$$|\{\text{subsets of } S\}| = |\{\text{subsets of } [n]\}|.$$

In other words,

$$(\text{\# of subsets of } S) = (\text{\# of subsets of } [n]) = 2^n$$

(by Theorem 4.7.2). □

4.7.3. Counting all k-element subsets

We move on to counting subsets of a given size.

Theorem 4.3.3 in Lecture 13 says:

Theorem 4.7.4. Let $n \in \mathbb{N}$, and let k be any number (not necessarily an integer). Let S be an n-element set. Then,

$$(\text{\# of } k\text{-element subsets of } S) = \binom{n}{k}.$$

Rigorous proof. We induct on n (without fixing k). That is, we use induction on n to prove the statement

$$P(n) := \left(\begin{array}{c} \text{“for any number } k \text{ and any } n\text{-element set } S, \\ \text{we have } (\text{\# of } k\text{-element subsets of } S) = \dbinom{n}{k}\text{”} \end{array} \right)$$

for each $n \in \mathbb{N}$.

Base case: Let k be any number. The only 0-element set is $\varnothing$, and its only subset is $\varnothing$. Thus, a 0-element set S necessarily has one 0-element subset ($\varnothing$) and no other subsets. Hence, it satisfies

$$(\text{\# of } k\text{-element subsets of } S) = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{else.} \end{cases}$$

However, we also have

$$\binom{0}{k} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{else} \end{cases}$$

(this follows easily from the definition of binomial coefficients). By comparing these two equalities, we see that any 0-element set S satisfies

$$(\text{\# of } k\text{-element subsets of } S) = \binom{0}{k}.$$

In other words, $P(0)$ holds.

Induction step: Let n be a positive integer. Assume (as the induction hypothesis) that $P(n-1)$ holds. We must prove that $P(n)$ holds.

So we consider any number k and any n-element set S. We must prove that

$$(\text{\# of } k\text{-element subsets of } S) = \binom{n}{k}.$$

We rename the n elements of S as $1, 2, \ldots, n$ (this corresponds formally to constructing a bijection $\alpha : S \to [n]$ and applying it elementwise to subsets of S, as we did in the proof of Theorem 4.7.3), so we must prove that

$$(\text{\# of } k\text{-element subsets of } [n]) = \binom{n}{k}.$$

To prove this, we define

- a **red set** to be a k-element subset of $[n]$ that contains n;
- a **green set** to be a k-element subset of $[n]$ that does not contain n.

For instance, for $n = 4$ and $k = 2$, the red sets are

$$\{1,4\}, \qquad \{2,4\}, \qquad \{3,4\},$$

while the green sets are

$$\{1,2\}, \qquad \{1,3\}, \qquad \{2,3\}.$$

Each k-element subset of $[n]$ is either red or green (but not both). Hence, using the sum rule for two sets, we find

$$\begin{aligned} &(\text{\# of } k\text{-element subsets of } [n]) \\ &= (\text{\# of red sets}) + (\text{\# of green sets}). \end{aligned} \tag{6}$$

(This is proved just as we proved (5) in the rigorous proof of Theorem 4.7.2.)

The green sets are just the k-element subsets of $[n-1]$. Thus,

$$\begin{aligned} (\text{\# of green sets}) &= (\text{\# of } k\text{-element subsets of } [n-1]) \\ &= \binom{n-1}{k} \end{aligned}$$

(by the statement $P(n-1)$, which we have assumed to hold).

Now, let's try to count the red sets.

Let us refer to the $(k-1)$-element subsets of $[n-1]$ as **blue sets**. If R is a red set, then $R \setminus \{n\}$ is a blue set[6]. Thus, we obtain a map

$$\begin{aligned} \text{rem}_n : \{\text{red sets}\} &\to \{\text{blue sets}\}, \\ R &\mapsto R \setminus \{n\}. \end{aligned}$$

Conversely, if B is a blue set, then $B \cup \{n\}$ is a red set[7]. Thus, we obtain a map

$$\begin{aligned} \text{ins}_n : \{\text{blue sets}\} &\to \{\text{red sets}\}, \\ B &\mapsto B \cup \{n\}. \end{aligned}$$

These two maps rem_n and ins_n are mutually inverse[8]. Thus, the map rem_n has an inverse, i.e., is bijective (by Theorem 4.5.7 in Lecture 15). Hence, we have found a bijection from $\{\text{red sets}\}$ to $\{\text{blue sets}\}$ (namely, rem_n). The bijection principle therefore yields

$$|\{\text{red sets}\}| = |\{\text{blue sets}\}|.$$

In other words,

$$\begin{aligned} (\text{\# of red sets}) &= (\text{\# of blue sets}) \\ &= (\text{\# of } (k-1)\text{-element subsets of } [n-1]) \\ &\qquad \text{(since this is how the blue sets were defined)} \\ &= \binom{n-1}{k-1} \end{aligned}$$

(again by the statement $P(n-1)$, but now applied to $k-1$ instead of k). Note that we deliberately formulated $P(n)$ as a "for any k" statement (rather than fixing k at the onset of our proof), so that we were now able to apply $P(n-1)$ to $k-1$ instead of k.

[6] *Proof.* Let R be a red set. Then, R is a k-element set (by the definition of a red set), so that $|R| = k$. Moreover, $n \in R$ (by the definition of a red set), so that $\{n\} \subseteq R$. Hence, the difference rule (Theorem 4.6.7 **(b)**, applied to $S = R$ and $T = \{n\}$) yields $|R \setminus \{n\}| = \underbrace{|R|}_{=k} - \underbrace{|\{n\}|}_{=1} = k-1$. Hence, $R \setminus \{n\}$ is a $(k-1)$-element set. Since $R \setminus \{n\}$ is furthermore a subset of $[n-1]$ (because R is a subset of $[n]$, and we are removing n from it), we thus conclude that $R \setminus \{n\}$ is a $(k-1)$-element subset of $[n-1]$, that is, a blue set.

[7] *Proof.* Let B be a blue set. Then, B is a $(k-1)$-element subset of $[n-1]$ (by the definition of "blue set"). In other words, $|B| = k-1$ and $B \subseteq [n-1]$. From $B \subseteq [n-1]$, we obtain $n \notin B$ (since $n \notin [n-1]$). Hence, Theorem 4.6.4 (applied to $S = B$ and $t = n$) yields $|B \cup \{n\}| = |B| + 1 = k$ (since $|B| = k-1$). Hence, $B \cup \{n\}$ is a k-element set. Furthermore, $B \cup \{n\}$ is a subset of $[n]$ (since $B \subseteq [n-1] \subseteq [n]$ and $\{n\} \subseteq [n]$) that contains n (since $n \in \{n\} \subseteq B \cup \{n\}$). Thus, $B \cup \{n\}$ is a k-element subset of $[n]$ that contains n. In other words, $B \cup \{n\}$ is a red set (by the definition of "red set").

[8] This can be proved just as in our above proof of Theorem 4.7.2.

Now, (6) becomes

$$(\# \text{ of } k\text{-element subsets of } [n]) = \underbrace{(\# \text{ of red sets})}_{=\binom{n-1}{k-1}} + \underbrace{(\# \text{ of green sets})}_{=\binom{n-1}{k}}$$
$$= \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

by Pascal's recurrence (Theorem 2.5.3 in Lecture 6). But this is precisely the equality that we have to prove. This completes the induction step, and thus Theorem 4.7.4 is proved. □

Remark 4.7.5. Our above proof of Theorem 4.7.4 can be simplified: There is no need to "rename" the elements of S as $1, 2, \ldots, n$ in the induction step. Instead, we could have just as well picked an arbitrary element t of S (such an element exists, since $|S| = n > 0$ entails that S is nonempty) and defined

- a **red set** to be a k-element subset of S that contains t;
- a **green set** to be a k-element subset of S that does not contain t.

Then, a simple application of Theorem 4.6.7 **(b)** would have shown that $S \setminus \{t\}$ is an $(n-1)$-element set, so we could apply our induction hypothesis $P(n-1)$ to it. Thus, the above argument could be made using S, t and $S \setminus \{t\}$ instead of $[n]$, n and $[n-1]$. In particular, the green sets would be precisely the k-element subsets of $S \setminus \{t\}$, whereas the red sets would be in one-to-one correspondence (i.e., bijection) with the $(k-1)$-element subsets of $S \setminus \{t\}$ (and the bijection would be given by removing/inserting t). This argument would be not only shorter but also more conceptual than the one we gave above.

However, I chose to give the proof I gave because it has the advantage of familiarity (the set $[n] = \{1, 2, \ldots, n\}$ is easier to visualize than an arbitrary n-element set), and in order to illustrate how the bijection principle can be used to rename the elements of a given set in a convenient way.

Likewise, Theorem 4.7.3 could also be proved more directly: Instead of deducing it from Theorem 4.7.2 via "renaming", we could have proved it by induction, again picking an element t of S in the induction step, defining red and green sets, and counting both kinds of sets using the induction hypothesis (applied to the $(n-1)$-element set $S \setminus \{t\}$).

Let us derive a nice, if simple, corollary from our last few theorems:

Corollary 4.7.6. Let $n \in \mathbb{N}$. Then,

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof. Consider the n-element set $[n] = \{1, 2, \ldots, n\}$. This set has size n, so each subset of $[n]$ must have size $\leq n$ (by Theorem 4.6.7 **(a)**). Hence, each subset of $[n]$ has size 0 or size 1 or size 2 or $\cdots$ or size n. Thus, we can write the set

$$\{\text{subsets of } [n]\}$$

as a union

$$\begin{aligned}
&\{\text{0-element subsets of } [n]\} \\
&\qquad \cup \{\text{1-element subsets of } [n]\} \\
&\qquad \cup \{\text{2-element subsets of } [n]\} \\
&\qquad \cup \cdots \\
&\qquad \cup \{n\text{-element subsets of } [n]\} .
\end{aligned}$$

Furthermore, this union is a union of disjoint sets (since a subset of $[n]$ cannot have several distinct sizes at once). Therefore, the sum rule for k sets (Theorem 4.6.6) yields

$$\begin{aligned}
&|\{\text{subsets of } [n]\}| \\
&= |\{\text{0-element subsets of } [n]\}| \\
&\qquad + |\{\text{1-element subsets of } [n]\}| \\
&\qquad + |\{\text{2-element subsets of } [n]\}| \\
&\qquad + \cdots \\
&\qquad + |\{n\text{-element subsets of } [n]\}| .
\end{aligned}$$

[9]

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[9] In more details:

The $n+1$ sets

$$\begin{aligned}
&\{\text{0-element subsets of } [n]\} , \\
&\{\text{1-element subsets of } [n]\} , \\
&\{\text{2-element subsets of } [n]\} , \\
&\ldots , \\
&\{n\text{-element subsets of } [n]\}
\end{aligned}$$

are finite (since $[n]$ has only finitely many subsets) and disjoint (since a subset of $[n]$ cannot have several distinct sizes at once). Thus, the sum rule for k sets (Theorem 4.6.6) yields

$$\begin{aligned}
&\Big| \{0\text{-element subsets of } [n]\} \cup \{1\text{-element subsets of } [n]\} \\
&\qquad \{2\text{-element subsets of } [n]\} \cup \cdots \cup \{n\text{-element subsets of } [n]\} \Big| \\
&= |\{0\text{-element subsets of } [n]\}| \\
&\qquad + |\{1\text{-element subsets of } [n]\}| \\
&\qquad + |\{2\text{-element subsets of } [n]\}| \\
&\qquad + \cdots \\
&\qquad + |\{n\text{-element subsets of } [n]\}| .
\end{aligned}$$

Since

$$\{\text{subsets of } [n]\}$$

is the union

$$\begin{aligned}
&\{0\text{-element subsets of } [n]\} \\
&\qquad \cup \{1\text{-element subsets of } [n]\} \\
&\qquad \cup \{2\text{-element subsets of } [n]\} \\
&\qquad \cup \cdots \\
&\qquad \cup \{n\text{-element subsets of } [n]\} ,
\end{aligned}$$

we can rewrite this equality as

$$\begin{aligned}
&|\{\text{subsets of } [n]\}| \\
&= |\{0\text{-element subsets of } [n]\}| \\
&\qquad + |\{1\text{-element subsets of } [n]\}| \\
&\qquad + |\{2\text{-element subsets of } [n]\}| \\
&\qquad + \cdots \\
&\qquad + |\{n\text{-element subsets of } [n]\}| .
\end{aligned}$$

In other words,

$$\begin{aligned}
&(\text{\# of subsets of } [n]) \\
&= (\text{\# of 0-element subsets of } [n]) \\
&\qquad + (\text{\# of 1-element subsets of } [n]) \\
&\qquad + (\text{\# of 2-element subsets of } [n]) \\
&\qquad + \cdots \\
&\qquad + (\text{\# of } n\text{-element subsets of } [n]) \\
&= \sum_{k=0}^{n} \underbrace{(\text{\# of } k\text{-element subsets of } [n])}_{\substack{=\binom{n}{k} \\ (\text{by Theorem 4.7.4, applied to } S=[n])}} \\
&= \sum_{k=0}^{n} \binom{n}{k}.
\end{aligned}$$

Thus,

$$\sum_{k=0}^{n} \binom{n}{k} = (\text{\# of subsets of } [n]) = 2^n$$

(by Theorem 4.7.2). □

Corollary 4.7.6 can also be easily obtained from the binomial formula (exercise!).

4.7.4. Recounting pairs

Proposition 4.4.3 in Lecture 13 says:

> **Proposition 4.7.7.** Let $n \in \mathbb{N}$. Then:
> **(a)** The # of pairs (a, b) with $a, b \in [n]$ is n^2.
> **(b)** The # of pairs (a, b) with $a, b \in [n]$ and $a < b$ is $1 + 2 + \cdots + (n-1)$.
> **(c)** The # of pairs (a, b) with $a, b \in [n]$ and $a = b$ is n.
> **(d)** The # of pairs (a, b) with $a, b \in [n]$ and $a > b$ is $1 + 2 + \cdots + (n-1)$.

Let us reprove part **(b)** of this proposition rigorously:

Rigorous proof of Proposition 4.7.7 ***(b)*** *(sketched).* If (a, b) is a pair with $a, b \in [n]$ and $a < b$, then the first entry of this pair (that is, the number a) must be one of the numbers $1, 2, \ldots, n-1$ (because $a < b \leq n$ forces a to be $\leq n-1$). Thus, by the sum rule for k sets (Theorem 4.6.6), we have

$$\begin{aligned}
&(\text{\# of pairs } (a,b) \text{ with } a,b \in [n] \text{ and } a<b) \\
&= \sum_{k=1}^{n-1} \underbrace{(\text{\# of pairs } (a,b) \text{ with } a,b \in [n] \text{ and } a<b \text{ and } a=k)}_{\substack{=n-k \\ \text{(because these pairs are } (k,\ k+1),\ (k,\ k+2),\ \ldots,\ (k,n)) \\ \text{(strictly speaking, this argument is an application of} \\ \text{the bijection principle))}}} \\
&= \sum_{k=1}^{n-1} (n-k) = (n-1) + (n-2) + \cdots + (n-(n-1)) \\
&= (n-1) + (n-2) + \cdots + 1 \\
&= 1 + 2 + \cdots + (n-1),
\end{aligned}$$

and thus Proposition 4.7.7 **(b)** is proven. □

4.8. Where do we stand now?

Recall the introductory counting problems from the start of Chapter 4 (Lecture 12). We can now answer some of these:

- How many ways are there to choose 3 odd integers between 0 and 20, if the order matters (i.e., we count the choice $1,3,5$ as different from the choice $3,1,5$)? (The answer is 1000.)

  We can solve this now: To choose 3 odd integers between 0 and 20, if the order matters, amounts to choosing a 3-tuple (a,b,c) where $a,b,c \in \{1,3,5,\ldots,19\}$. Since this set $\{1,3,5,\ldots,19\}$ is a 10-element set (because Proposition 4.2.1 from Lecture 12 yields that the # of odd integers between 0 and 20 is $(20+1)\,//2 = 10$), the # of these 3-tuples is $10 \cdot 10 \cdot 10 = 1000$ (by Theorem 4.4.5 in Lecture 13).

- How many ways are there to choose 3 odd integers between 0 and 20, if the order does not matter? (The answer is 220.)

  We cannot solve this yet, at least not if the values 3 and 20 are generalized to k and n. This will be done next week.

- How many ways are there to choose 3 distinct odd integers between 0 and 20, if the order matters? (The answer is 720.)

  We cannot solve this yet, at least not if the values 3 and 20 are generalized to k and n. This will be done next week.

- How many ways are there to choose 3 distinct odd integers between 0 and 20, if the order does not matter? (The answer is 120.)

We can solve this now: This amounts to counting the 3-element subsets of $\{1,3,5,\ldots,19\}$; but Theorem 4.7.4 answers such questions. Since the set $\{1,3,5,\ldots,19\}$ has size 10, its number of 3-element subsets is $\binom{10}{3} = \frac{10 \cdot 9 \cdot 8}{3!} = 120$.

- How many prime factorizations does 200 have (where we count different orderings as distinct)? (The answer is 10. This is a mix between a number theory problem and a counting problem.)

  We can solve this now, at least for 200: We know that $200 = 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5$. Thus, by the fundamental theorem of arithmetic, all prime factorizations of 200 consist of five factors, three of which are 2's and two of which are 5's. The only freedom is in choosing where to place the three 2's among the five positions (of course, the two 5's will then have to occupy the remaining positions). There are 5 factors in total, so 5 positions, and we have to choose 3 of these 5 positions to put our three 2's in. This is tantamount to choosing a 3-element subset of $[5]$ (the subset of the positions in which we put the 2's), and the # of ways to do this is $\binom{5}{3} = 10$ (by Theorem 4.7.4). Thus, 200 has 10 prime factorizations (if we count different orderings as distinct).

  However, it is trickier to extend this reasoning to prime factorizations of 150. Indeed, $150 = 2 \cdot 3 \cdot 5 \cdot 5$, so a prime factorization of 150 has one 2, one 3 and two 5's. How many ways are there to place one 2, one 3 and two 5's in altogether four positions? I'll leave this one to you for now, but we will come back to this later.

- How many ways are there to tile a 2×15-rectangle with dominos (i.e., rectangles of size 1×2 or 2×1) ? (The answer is 987.)

  We cannot solve this yet.

- How many addends do you get when you expand the product $(a+b)(c+d+e)(f+g)$? (The answer is 12.)

  We can solve this now: Each addend consists of exactly one of a and b, exactly one of c, d and e, and exactly one of f and g. So the addends are in one-to-one correspondence with the triples (x,y,z) where $x \in \{a,b\}$ and $y \in \{c,d,e\}$ and $z \in \{f,g\}$. Thus, their # is $2 \cdot 3 \cdot 2$ (since $\{a,b\}$ is a 2-element set, $\{c,d,e\}$ is a 3-element set, and $\{f,g\}$ is a 2-element set).

  Note that we are using the fact that the addends all end up distinct, so they don't cancel or combine.

- How many different monomials do you get when you expand the product $(a-b)\left(a^2+ab+b^2\right)$? (This one is more of an algebra problem, but I wanted to list it because it is connected to counting. The answer is 2, because $(a-b)\left(a^2+ab+b^2\right)=a^3-b^3$.)

This is not a combinatorics problem: The answer is 2, because we have $(a-b)\left(a^2+ab+b^2\right)=a^3-b^3$. The other addends all cancel out, so you get an answer much less than 6.

In general, problems like this (where you count addends after cancellation and combination) cannot be solved combinatorially; you have to actually expand and collect.

- How many positive divisors does 24 have? (We can actually list them: 1, 2, 3, 4, 6, 8, 12, 24. This one is again a mix of a counting problem and a number theory problem.)

  We cannot solve this yet.

References

[Math222] Darij Grinberg, *Math 222: Enumerative Combinatorics, Fall 2022.* `https://www.cip.ifi.lmu.de/~grinberg/t/22fco/`