# Math 221 Winter 2023, Lecture 14: Enumeration

**website:** `https://www.cip.ifi.lmu.de/~grinberg/t/23wd`

# 4. An introduction to enumeration

## 4.5. Maps (aka functions)

### 4.5.1. Functions, informally

One of the main notions in mathematics is that of a **function**, aka **map**, aka **mapping**, aka **transformation**.

Intuitively, a function is a "black box" that takes inputs and transforms them into outputs. For example, the "$f(t) = t^2$" function takes a real number $t$ and outputs its square $t^2$.

You can thus think of a function as a rule for producing an output from an input. This gives the following **provisional** definition of a function:

> **Definition 4.5.1** (Informal definition of a function)**.** Let $X$ and $Y$ be two sets. A **function** from $X$ to $Y$ is (provisionally) a rule that transforms each element of $X$ into some element of $Y$.
>
> If this function is called $f$, then the result of applying it to a given $x \in X$ (that is, the output produced by $f$ when $x$ is the input) will be called $f(x)$ (or sometimes $fx$).

This is not a real definition, as it only kicks the can down the road: It defines "function" in terms of "rule", but what is a rule? But it gives some good intuition, provided that it is correctly understood. Here are some comments that should clarify it:

- A function has to "work" for each element of $X$. It cannot decline to operate on some elements! Thus, "take the reciprocal" is not a function from $\mathbb{R}$ to $\mathbb{R}$, since it does not operate on 0 (because 0 has no reciprocal). However, "take the reciprocal" is a function from $\mathbb{R} \setminus \{0\}$ to $\mathbb{R}$, since any nonzero real number does have a reciprocal.

- A function must not be ambiguous. Each input must produce exactly one output. Thus, "take your number to some random power" is not a function from $\mathbb{R}$ to $\mathbb{R}$, since different powers give different results. (There is a "multi-valued" variant of functions around, but they aren't called "functions".)

- We write "$f : X \to Y$" for "$f$ is a function from $X$ to $Y$".

- Instead of saying "$f(x) = y$", we can say "$f$ transforms $x$ into $y$" or "$f$ sends $x$ to $y$" or "$f$ maps $x$ to $y$" or "$f$ takes the value $y$ at $x$" or "$y$ is the value of $f$ at $x$" or "$y$ is the image of $x$ under $f$" or "applying $f$ to $x$ yields $y$" or "$f$ takes $x$ to $y$" or "$f : x \mapsto y$". All of these statements are synonyms.

  For instance, if $f$ is the "take the square" function from $\mathbb{R}$ to $\mathbb{R}$, then $f(2) = 2^2 = 4$, so that $f$ transforms 2 to 4, or sends 2 to 4, or takes the value 4 at 2, etc., or $f : 2 \mapsto 4$.

- As the above terminology suggests, the **value** of a function $f$ at an input $x$ means the corresponding output $f(x)$.

- The notation

  $$\begin{aligned} X &\to Y, \\ x &\mapsto (\text{some expression involving } x) \end{aligned}$$

  means "the function from $X$ to $Y$ that sends each element $x$ of $X$ to the expression on the right hand side". Here, the expression can (for example) be $x^2$ or $\dfrac{1}{x+4}$ or $\dfrac{x}{x+2}$.

  For example,

  $$\begin{aligned} \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto x^2 \end{aligned}$$

  is the "take the square" function (sending each element $x$ of $\mathbb{R}$ to $x^2$). For another example,

  $$\begin{aligned} \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto \frac{x}{\sin x + 15} \end{aligned}$$

  is the function that takes the sine of the input, then adds 15, then divides the input by the result. (Note that this is well-defined, since $\sin x + 15$ is never zero and thus the expression $\dfrac{x}{\sin x + 15}$ is always meaningful, so we really get a function from $\mathbb{R}$ to $\mathbb{R}$.)

  For yet another example,

  $$\begin{aligned} \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto 2 \end{aligned}$$

  is the function that sends each real number $x$ to 2; this is an example of a constant function. (This is a case where our "expression involving $x$"

does not actually contain $x$. This is perfectly fine; it's just a very simple particular case.)

For yet another example,

$$\begin{aligned} \mathbb{Z} &\to \mathbb{Q}, \\ x &\mapsto 2^x \end{aligned}$$

is a function. Some of its values are listed in the following table:

| $x$ | $-2$ | $-1$ | $0$ | $1$ | $2$ |
|---|---|---|---|---|---|
| $2^x$ | $\frac{1}{4}$ | $\frac{1}{2}$ | $1$ | $2$ | $4$ |

- The notation

$$\begin{aligned} f : X &\to Y, \\ x &\mapsto (\text{some expression involving } x) \end{aligned}$$

means that we take the function from $X$ to $Y$ that sends each $x \in X$ to the expression on the right hand side, and we call this function $f$.

(Or, if a function named $f$ has already been defined, this notation means that this $f$ **is** the function from $X$ to $Y$ that sends each $x \in X$ to the expression on the right hand side.)

For example, if we write

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R}, \\ x &\mapsto x^2 + 1, \end{aligned}$$

then $f$ henceforth will denote the function from $\mathbb{R}$ to $\mathbb{R}$ that sends each $x \in \mathbb{R}$ to $x^2 + 1$.

- If the set $X$ is finite, then a function $f : X \to Y$ can be specified by simply listing all its values. For example, I can define a function $h : \{0, 2, 4\} \to \mathbb{N}$ by setting

$$\begin{aligned} h(0) &= 92, \\ h(2) &= 20, \\ h(4) &= 92. \end{aligned}$$

The values here have been chosen at whim, for no particular reason. A function does not have to be "natural" or "meaningful" in any way; all it has to do is transform each element of $X$ into some element of $Y$.

- If $f$ is a function from $X$ to $Y$, then the sets $X$ and $Y$ are part of the function. Thus,

$$g_1 : \mathbb{Z} \to \mathbb{Q}, \\ x \mapsto 2^x$$

and

$$g_2 : \mathbb{N} \to \mathbb{Q}, \\ x \mapsto 2^x$$

and

$$g_3 : \mathbb{N} \to \mathbb{N}, \\ x \mapsto 2^x$$

are three distinct functions! We distinguish between them, so that we can later speak of the "domain" and the "target" of a function. Namely, the **domain** of a function $f : X \to Y$ is defined to be the set $X$, whereas the **target** of a function $f : X \to Y$ is defined to be the set $Y$. Thus, the above function $g_2$ has target $\mathbb{Q}$, whereas the function $g_3$ has target $\mathbb{N}$.

- When are two functions equal? In programming, functions are often understood to be (implemented) algorithms, and two algorithms can be different even if they compute the same thing. In mathematics, it's different: Only the domain, the target and the output values matter; the way they are computed does not (and indeed there might not even be a way to compute them). Two algorithms that (always) compute the same thing count for one function only.

  So when are two functions considered to be equal?

  Two functions $f_1 : X_1 \to Y_1$ and $f_2 : X_2 \to Y_2$ are said to be **equal** if and only if

$$X_1 = X_2 \qquad \text{and} \qquad Y_1 = Y_2 \qquad \text{and} \\ f_1(x) = f_2(x) \qquad \text{for all } x \in X_1.$$

  An example of two equal functions is

$$f_1 : \mathbb{R} \to \mathbb{R}, \\ x \mapsto x^2$$

and

$$f_2 : \mathbb{R} \to \mathbb{R}, \\ x \mapsto |x|^2,$$

since each $x \in \mathbb{R}$ satisfies $x^2 = |x|^2$.

At this point, we have a good idea of what a function is, but the provisional definition given above (Definition 4.5.1) wasn't as precise as we would like. Even worse, the word "rule" in that definition is still unclear, and prevents us from dealing with functions that can neither be given by an explicit formula (such as "take the square") nor be specified by a complete list of values (e.g., since the domain is infinite). Thus, we need a better definition of a function.

This is what we will do today. The trick is to first define the more general concept of a **relation**, and then to characterize functions as relations with a certain property.

### 4.5.2. Relations

**Relations** (to be specific: binary relations) are another concept that you have already seen on myriad examples:

- The relation $\subseteq$ is a relation between two sets. For example, $\{1,3\} \subseteq \{1,2,3,4\}$ but $\{1,5\} \not\subseteq \{1,2,3,4\}$.
- The order relations $\leq$ and $<$ and $>$ and $\geq$ are relations between two integers (or rational numbers, or real numbers). For example, $1 \leq 5$ but $1 \not\leq -1$.
- The containment relation $\in$ is a relation between an object and a set. For instance, $3 \in \{1,2,3,4\}$ but $5 \notin \{1,2,3,4\}$.
- The divisibility relation $\mid$ is a relation between two integers.
- The relation "coprime" is a relation between two integers.
- In plane geometry, there are lots of relations: "parallel" (between two lines), "perpendicular" (between two lines), "congruent" (between two shapes), "similar", "directly similar", etc.
- For any given integer $n$, the relation "congruent modulo $n$" is a relation between two integers. Let me call it $\overset{n}{\equiv}$. Thus, $a \overset{n}{\equiv} b$ holds if and only if $a \equiv b \mod n$. For example, $2 \overset{3}{\equiv} 8$ but $2 \overset{3}{\not\equiv} 7$.

What do these relations all have in common? They can be applied to pairs of objects. Applying a relation to a pair of objects gives a statement that can be true or false. For example, applying the relation "coprime" to the pair $(5,8)$ yields the statement "5 is coprime to 8", which is true. Applying it to the pair $(5,10)$ yields the statement "5 is coprime to 10", which is false.

A general relation $R$ relates elements of a set $X$ with elements of a set $Y$. For any pair $(x,y) \in X \times Y$ (that is, for any pair consisting of an element $x \in X$ and an element $y \in Y$), we can apply the relation $R$ to the pair $(x,y)$, obtaining a statement "$x \ R \ y$" that is either true or false. To describe this relation $R$, we need to know which pairs $(x, y) \in X \times Y$ do satisfy $x \; R \; y$ and which pairs don't. In other words, we need to know the **set** of all pairs $(x, y) \in X \times Y$ that satisfy $x \; R \; y$. For a rigorous definition of a relation, we simply take the relation $R$ to **be** this set of pairs. In other words, we define relations as follows:

**Definition 4.5.2.** Let $X$ and $Y$ be two sets. A **relation** from $X$ to $Y$ is a subset of $X \times Y$.

If $R$ is a relation from $X$ to $Y$, and if $(x, y) \in X \times Y$ is any pair, then

- we write $x \; R \; y$ if $(x, y) \in R$;
- we write $x \not R \; y$ if $(x, y) \notin R$.

All the relations we have seen so far can be recast in terms of this definition:

- The divisibility relation $\mid$ is a subset of $\mathbb{Z} \times \mathbb{Z}$, namely the subset

$$\begin{aligned} & \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \text{ divides } y\} \\ & = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid \text{there exists some } z \in \mathbb{Z} \text{ such that } y = xz\} \\ & = \{(x, xz) \mid x \in \mathbb{Z} \text{ and } z \in \mathbb{Z}\}. \end{aligned}$$

  For instance, the pairs $(2, 4)$ and $(3, 9)$ and $(10, 20)$ belong to this subset, whereas the pairs $(2, 3)$ and $(2, 15)$ and $(10, 5)$ do not.

- The coprimality relation ("coprime to") is a subset of $\mathbb{Z} \times \mathbb{Z}$, namely the subset

$$\begin{aligned} & \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \text{ is coprime to } y\} \\ & = \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid \gcd(x, y) = 1\}. \end{aligned}$$

  It contains, for instance, $(2, 3)$ and $(7, 9)$, but not $(4, 6)$.

- For any $n \in \mathbb{Z}$, the "congruent modulo $n$" relation $\overset{n}{\equiv}$ is a subset of $\mathbb{Z} \times \mathbb{Z}$, namely the subset

$$\begin{aligned} & \{(x, y) \in \mathbb{Z} \times \mathbb{Z} \mid x \equiv y \bmod n\} \\ & = \{(x, \; x + nz) \mid x \in \mathbb{Z} \text{ and } z \in \mathbb{Z}\}. \end{aligned}$$

- A geometric example: Let $P$ be the set of all points in the plane, and let $L$ be the set of all lines in the plane. Then, the "lies on" relation (as in "a point lies on a line") is a subset of $P \times L$, namely the subset

$$\{(p, \ell) \in P \times L \mid \text{the point } p \text{ lies on the line } \ell\}.$$

- If $A$ is any set, then the **equality relation** on $A$ is the subset $E_A$ of $A \times A$ given by

$$\begin{aligned} E_A &= \{(x,y) \in A \times A \mid x = y\} \\ &= \{(x,x) \mid x \in A\}. \end{aligned}$$

  Two elements $x$ and $y$ of $A$ satisfy $x \; E_A \; y$ if and only if they are equal.

- We can literally take any subset of $X \times Y$ (where $X$ and $Y$ are two sets) and it will be a relation from $X$ to $Y$. Just as with functions, a relation does not have to follow any "meaningful" rule. For example, here is a relation from $\{1,2,3\}$ to $\{5,6,7\}$:

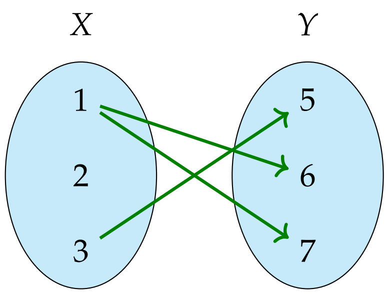
$$\{(1,6), (1,7), (3,5)\}.$$

  Equivalently, it can be specified by the table

| | 5 | 6 | 7 |
|---|---|---|---|
| 1 | no | yes | yes |
| 2 | no | no | no |
| 3 | yes | no | no |

  (where a "yes" in row $x$ and column $y$ means that $(x,y)$ belongs to the relation). If we call this relation $R$, then we have $1 \; R \; 6$ and $1 \; R \; 7$ and $3 \; R \; 5$ but not $1 \; R \; 5$ or $2 \; R \; 6$.

A good way to visualize a relation $R$ from a set $X$ to a set $Y$ (at least when $X$ and $Y$ are finite) is by drawing the sets $X$ and $Y$ as blobs, drawing their elements as nodes within these blobs, and drawing an arrow from the $x$-node to the $y$-node for every pair $(x,y)$ that belongs to the relation $R$. For example, the relation $R$ in our last example can be visualized as follows:




(1)

#### 4.5.3. Functions, formally

We can now define functions rigorously:

**Definition 4.5.3** (Rigorous definition of a function)**.** Let $X$ and $Y$ be two sets. A **function** from $X$ to $Y$ means a relation $R$ from $X$ to $Y$ that has the following property:

- **Output uniqueness:** For each $x \in X$, there exists **exactly one** $y \in Y$ such that $x \ R \ y$.

If $R$ is a function from $X$ to $Y$, and if $x$ is an element of $X$, then the unique element $y \in Y$ satisfying $x \ R \ y$ will be called $f(x)$.

In our above example, the relation

$$\{(1,6),\ (1,7),\ (3,5)\}$$

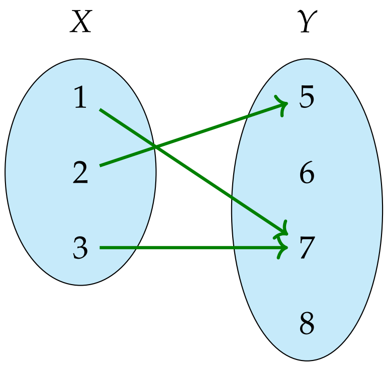
(which we illustrated in (1)) is not a function from $\{1,2,3\}$ to $\{5,6,7\}$. In fact, it violates output uniqueness at $x = 1$ (since there are two $y \in \{5,6,7\}$ that satisfy $1 \ R \ y$) and also violates it at $x = 2$ (since there are no $y \in \{5,6,7\}$ that satisfy $2 \ R \ y$). Each of these two violations is reason enough to disqualify this relation from being a function.

In our above list of relations, only the equality relation $E_A$ is a function.

Here is an example of a function from $X = \{1,2,3\}$ to $Y = \{5,6,7,8\}$: the relation

$$\{(1,7),\ (2,5),\ (3,7)\}.$$

This relation satisfies output uniqueness and thus is a function. Visualized by blobs and arrows, it looks as follows:




If we denote this function by $f$, then $f(1) = 7$ and $f(2) = 5$ and $f(3) = 7$.

Our way of visualizing relations by blobs and arrows makes the output uniqueness property quite intuitive: This property just says that for each $x \in X$, there is **exactly one** arrow starting at the $x$-node. In other words, each node in the $X$-blob has to be the starting point of exactly one arrow. Thus, a function is a relation whose visual picture has exactly one arrow coming out of each $X$-node.

Now we have two definitions of a function: the provisional definition (Definition 4.5.1) and the rigorous one (Definition 4.5.3). These two definitions are equivalent. Indeed:

- If $R$ is a function from $X$ to $Y$ in the sense of the rigorous definition (i.e., a relation from $X$ to $Y$ that satisfies output uniqueness), then $R$ can also be viewed as a rule that sends each element $x$ of $X$ to some element of $Y$ (namely, to the unique $y \in Y$ that satisfies $x \ R \ y$). Thus, $R$ becomes a function in the provisional sense.

- Conversely, if $f$ is a function from $X$ to $Y$ in the provisional sense (i.e., a rule sending elements of $X$ to elements of $Y$), then $f$ can also be viewed as a function in the rigorous sense (i.e., as a relation from $X$ to $Y$ that satisfies output uniqueness), as follows: Let $R$ be the set

$$\{(x, f(x)) \mid x \in X\}.$$

  This set $R$ is a subset of $X \times Y$, that is, a relation from $X$ to $Y$. (In a more intuitive language, this relation $R$ is characterized as follows: Two elements $x \in X$ and $y \in Y$ satisfy $x \ R \ y$ if and only if $y = f(x)$. That is, roughly speaking, the relation $R$ relates each input $x \in X$ with the corresponding output value $f(x) \in Y$ and with nothing else.) This relation $R$ satisfies output uniqueness (because each input $x \in X$ produces exactly one output value $f(x)$), and therefore is a function from $X$ to $Y$ in the rigorous sense. Thus, $f$ becomes a function in the rigorous sense (namely, the rigorous function $R$).

Therefore, we can translate rigorous functions into provisional ones and vice versa. We thus shall think of the two concepts as being the same. In particular, all the notations we have introduced for provisional functions will be used for rigorous ones.

### 4.5.4. Some more examples of functions

Let us give some examples of functions as well as some examples of what looks like functions but are not.

**Example 4.5.4.** Consider the function

$$f_0 : \{1, 2, 3, 4\} \to \{1, 2, 3, 4\}$$

that sends $1, 2, 3, 4$ to $3, 2, 3, 3$, respectively. As a rigorous function, it is the relation $R$ that satisfies

$$1 \ R \ 3, \qquad 2 \ R \ 2, \qquad 3 \ R \ 3, \qquad 4 \ R \ 3$$

(and nothing else). In other words, it is the relation

$$\{(1,3), (2,2), (3,3), (4,3)\}.$$

**Example 4.5.5.** What about the function

$$f_1 : \{1, 2, 3, 4\} \to \{1, 2, 3\}, \\ n \mapsto n \quad ?$$

Such a function $f_1$ does not exist, since it would have to send 4 to 4, but 4 is not in the target $\{1, 2, 3\}$.

This is a pedantic issue, but it should be kept in mind: Not every expression that appears to define a function actually defines a function. Make sure that the expression to the right of the "$\mapsto$" symbol always is an actual element of the target (which, in this case, is the set $\{1, 2, 3\}$).

**Example 4.5.6.** Consider the function

$$f_2 : \{1, 2, 3, \ldots\} \to \{1, 2, 3, \ldots\}, \\ n \mapsto (\text{the number of positive divisors of } n).$$

As a relation, it is

$$\{(1,1), (2,2), (3,2), (4,3), (5,2), (6,4), (7,2), (8,4), (9,3), \ldots\}.$$

(We cannot list all the pairs, since there are infinitely many.) Thus, $f_2(1) = 1$ and $f_2(2) = 2$ and $f_2(3) = 2$ and so on.

**Example 4.5.7.** What about the function

$$\widetilde{f_2} : \mathbb{Z} \to \{1, 2, 3, \ldots\}, \\ n \mapsto (\text{the number of positive divisors of } n) \quad ?$$

There is no such function $\widetilde{f_2}$, since $\widetilde{f_2}(0)$ would have to be undefined or $\infty$ (because 0 has infinitely many positive divisors).

This is the exact same problem that we had with the non-function $f_1$ above.

**Example 4.5.8.** What about the function

$$f_3 : \{1, 2, 3, \ldots\} \to \{1, 2, 3, \ldots\}, \\ n \mapsto (\text{the smallest prime divisor of } n) \quad ?$$

Again, there is no such function $f_3$, since $f_3(1)$ makes no sense (indeed, the number 1 has no prime divisors, thus no smallest prime divisor).

This is essentially the same problem as with the function $\widetilde{f_2}$ from the previous example, except that this time the value $f_3(1)$ is really undefined (as opposed to just failing to belong to the target).

Note that the function $f_3$ "almost" exists: There is a relation "$y$ is the smallest prime divisor of $x$" from $\{1, 2, 3, \ldots\}$ to $\{1, 2, 3, \ldots\}$, but this relation fails the output uniqueness requirement at $x = 1$, and thus is not a function. However, we can make it into a function by removing the offending element 1 from its domain. That is, there is a function

$$\widetilde{f_3} : \{2, 3, 4, \ldots\} \to \{1, 2, 3, \ldots\},$$
$$n \mapsto (\text{the smallest prime divisor of } n).$$

**Example 4.5.9.** What about the function

$$f_4 : \mathbb{Q} \to \mathbb{Z},$$
$$\frac{a}{b} \mapsto a \qquad (\text{for } a, b \in \mathbb{Z}) \ ?$$

Restated in words, this is to be a function that takes a rational number as input, writes it as a ratio of two integers and outputs the numerator. Is there such a function?

Again, the answer is **no**. Again, the problem is a failure of output uniqueness, but this time, it fails not because the output does not exist (or does not belong to the target), but rather because the output is non-unique. For example, if $f_4$ was a function, then we would have the two equalities

$$f_4 (0.5) = f_4 \left(\frac{1}{2}\right) = 1 \qquad \text{and}$$
$$f_4 (0.5) = f_4 \left(\frac{3}{6}\right) = 3,$$

which contradict one another. The underlying issue is that a rational number can be written as a fraction in several different ways, and the numerators of these fractions will usually **not be the same**. Thus, if you follow the rule $\frac{a}{b} \mapsto a$ to compute the output of $f_4$ for a given input, your output will depend on how exactly you write your input as a fraction, and this is a violation of output uniqueness.

### 4.5.5. Well-definedness

The issues that we have seen in the last few examples (supposed functions failing to exist either because their output values make no sense, or because these values don't lie in $Y$, or because these values are ambiguous) are known as **well-definedness** issues. Often, mathematicians say that "a function is well-defined" when they mean that its definition does not suffer from such issues (i.e., its definition really defines a function). So you should read "This function is well-defined [or: not well-defined]" as "The definition we just gave really defines a function [or: does not actually define a function]".

For example, as we just saw, the function

$$f_4 : \mathbb{Q} \to \mathbb{Z}, \qquad \frac{a}{b} \mapsto a$$

is not well-defined (i.e., there is no such function), but the function

$$f_5 : \mathbb{Q} \to \mathbb{Q}, \qquad \frac{a}{b} \mapsto \frac{a^2}{b^2}$$

is well-defined (because if you write a given rational number as $\frac{a}{b}$ for different pairs $(a, b)$, the resulting quotients $\frac{a^2}{b^2}$ will all be equal). The function

$$f_1 : \{1, 2, 3, 4\} \to \{1, 2, 3\}, \qquad n \mapsto n$$

is not well-defined (since its supposed output $f_1(4)$ fails to lie in the target $\{1, 2, 3\}$), whereas the function

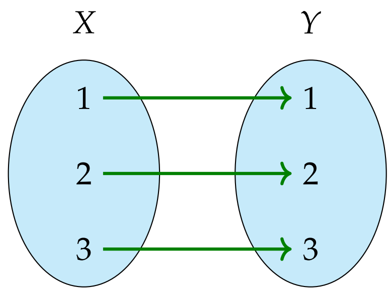
$$f_6 : \{1, 2, 3, 4\} \to \{1, 2, 3\}, \qquad n \mapsto 1 + (n\%3)$$

is well-defined (since its outputs at $1, 2, 3, 4$ are $2, 3, 1, 2$).

### 4.5.6. The identity function

**Definition 4.5.10.** For any set $A$, there is an **identity function** $\mathrm{id}_A : A \to A$. This is the function that sends each element $a \in A$ to $a$ itself. In other words, it is precisely the relation $E_A$ defined above.

Here is the blobs-and-arrows visualization of the identity function $\mathrm{id}_A$ for $A = \{1, 2, 3\}$:



### 4.5.7. Another example

As we said before, a function $f : X \to Y$ can be described either by a rule or by a list of values (if $X$ is finite) or as a relation. For instance, the "take the square" function on real numbers is the function

$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto x^2.$$

As a relation, it is the set

$$\left\{ \left( x, x^2 \right) \mid x \in \mathbb{R} \right\}.$$

### 4.5.8. Composition of functions

There are some ways to transform functions into other functions. The most important one is **composition**:

**Definition 4.5.11.** Let $X$, $Y$ and $Z$ be three sets. Let $f : Y \to Z$ and $g : X \to Y$ be two functions. Then, $f \circ g$ denotes the function

$$X \to Z, \quad x \mapsto f\left(g\left(x\right)\right).$$

In other words, $f \circ g$ is the function that first applies $g$ and then applies $f$. This function $f \circ g$ is called the **composition** of $f$ with $g$ (and I pronounce it "$f$ after $g$").

In terms of relations, if we view $f$ and $g$ as two relations $F$ and $G$ (as in Definition 4.5.3), then $f \circ g$ is the relation

$$\{(x, z) \mid \text{there exists } y \in Y \text{ such that } x\ G\ y \text{ and } y\ F\ z\} \text{ from } X \text{ to } Z.$$

**Example 4.5.12.** Let $\mathbb{R}_{\geq 0}$ denote the set of all nonnegative real numbers. Consider the two functions

$$f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad x \mapsto x^2$$

and

$$g : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad x \mapsto \frac{1}{x+7}.$$

Then, for any nonnegative real $x \in \mathbb{R}_{\geq 0}$, we have

$$(f \circ g)(x) = f(g(x)) = f\left(\frac{1}{x+7}\right) = \left(\frac{1}{x+7}\right)^2$$

whereas

$$(g \circ f)(x) = g(f(x)) = g\left(x^2\right) = \frac{1}{x^2+7}.$$

Note that these two results are different. Thus, $f \circ g \neq g \circ f$ in general.

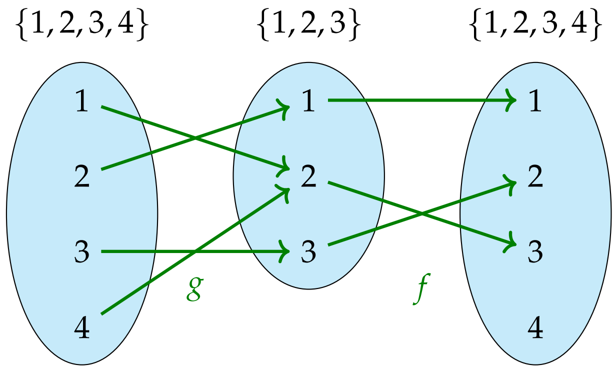
**Example 4.5.13.** Consider the two functions $f : \{1,2,3\} \to \{1,2,3,4\}$ and $g : \{1,2,3,4\} \to \{1,2,3\}$ given by the following tables of values:

| $i$ | 1 | 2 | 3 |
|---|---|---|---|
| $f(i)$ | 1 | 3 | 2 |

| $i$ | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| $g(i)$ | 2 | 1 | 3 | 2 |

.

These two functions can be visualized using blobs and arrows, and we can even reuse the target-blob from $g$ as the domain-blob for $f$:




.

This allows us to visually construct $f \circ g$ by removing the middle blob and merging each $g$-arrow with the $f$-arrow that starts where the $g$-arrow ends: