Math 221 Winter 2023, Lecture 13: Enumeration

website: `https://www.cip.ifi.lmu.de/~grinberg/t/23wd`

4. An introduction to enumeration

4.3. Counting subsets (cont'd)

4.3.1. Counting them all (cont'd)

Last time, we proved:[1]

Theorem 4.3.1. Let $n \in \mathbb{N}$. Then,

$$(\# \text{ of subsets of } \{1, 2, \ldots, n\}) = 2^n.$$

More generally, we have the following:

Theorem 4.3.2. Let $n \in \mathbb{N}$. Let S be an n-element set. Then,

$$(\# \text{ of subsets of } S) = 2^n.$$

Informal proof. This follows from Theorem 4.3.1, since we can rename the n elements of S as $1, 2, \ldots, n$. $\square$

For example,

$$(\# \text{ of subsets of } \{\text{"cat"}, \text{"dog"}, \text{"bat"}\}) = 2^3.$$

4.3.2. Counting the subsets of a given size

Let us now refine our question: Instead of counting all subsets of $\{1, 2, \ldots, n\}$, we shall only count the ones that have a given size k. Here, the **size** of a set means the # of its elements, i.e., how many distinct elements it has. (For example, the set $\{1, 4, 1, 15\}$ has size 3, nevermind that I needlessly listed one of its elements twice.) A set of size k is also known as a k**-element set**. (Soon we will define these concepts rigorously.)

For instance, $\{1, 2, 3, 4\}$ is a 4-element set. How many 2-element subsets does it have? It has six:

$$\{1, 2\}, \qquad \{1, 3\}, \qquad \{1, 4\}, \qquad \{2, 3\}, \qquad \{2, 4\}, \qquad \{3, 4\}.$$

[1] Recall that the symbol "#" means "number".

More generally, the answer to the question "how many k-element subsets does a given n-element set have" turns out to be the binomial coefficient $\dbinom{n}{k}$. Let us state this as a theorem and give an informal proof (which will easily become rigorous once we have the basic concepts of counting pinned down):[2]

Theorem 4.3.3. Let $n \in \mathbb{N}$, and let k be any number (not necessarily an integer). Let S be an n-element set. Then,

$$(\text{\# of } k\text{-element subsets of } S) = \binom{n}{k}.$$

Informal proof. We induct on n (without fixing k). That is, we use induction on n to prove the statement

$$P(n) := \left(\begin{array}{c} \text{“for any number } k \text{ and any } n\text{-element set } S, \\ \text{we have } (\text{\# of } k\text{-element subsets of } S) = \dbinom{n}{k}\text{”} \end{array} \right)$$

for each $n \in \mathbb{N}$.

Base case: Let k be any number. The only 0-element set is $\varnothing$, and its only subset is $\varnothing$. Thus, a 0-element set S necessarily has one 0-element subset ($\varnothing$) and no other subsets. Hence, it satisfies

$$(\text{\# of } k\text{-element subsets of } S) = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{else.} \end{cases}$$

However, we also have

$$\binom{0}{k} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{else} \end{cases}$$

(this follows easily from the definition of binomial coefficients). By comparing these two equalities, we see that any 0-element set S satisfies

$$(\text{\# of } k\text{-element subsets of } S) = \binom{0}{k}.$$

In other words, $P(0)$ holds.

Induction step: Let n be a positive integer. Assume (as the induction hypothesis) that $P(n-1)$ holds. We must prove that $P(n)$ holds.

So we consider any number k and any n-element set S. We must prove that

$$(\text{\# of } k\text{-element subsets of } S) = \binom{n}{k}.$$

[2] This theorem is exactly Theorem 2.5.12 in Lecture 6.

We rename the n elements of S as $1, 2, \ldots, n$, so we must prove that

$$(\text{\# of } k\text{-element subsets of } \{1, 2, \ldots, n\}) = \binom{n}{k}.$$

To prove this, we define

- a **red set** to be a k-element subset of $\{1, 2, \ldots, n\}$ that contains n;
- a **green set** to be a k-element subset of $\{1, 2, \ldots, n\}$ that does not contain n.

For instance, for $n = 4$ and $k = 2$, the red sets are

$$\{1, 4\}, \qquad \{2, 4\}, \qquad \{3, 4\},$$

while the green sets are

$$\{1, 2\}, \qquad \{1, 3\}, \qquad \{2, 3\}.$$

Each k-element subset of $\{1, 2, \ldots, n\}$ is either red or green (but not both). Hence,

$$\begin{aligned} &(\text{\# of } k\text{-element subsets of } \{1, 2, \ldots, n\}) \\ &= (\text{\# of red sets}) + (\text{\# of green sets}). \end{aligned} \tag{1}$$

The green sets are just the k-element subsets of $\{1, 2, \ldots, n-1\}$. Thus,

$$\begin{aligned} (\text{\# of green sets}) &= (\text{\# of } k\text{-element subsets of } \{1, 2, \ldots, n-1\}) \\ &= \binom{n-1}{k} \end{aligned}$$

(by the statement $P(n-1)$, which we have assumed to hold).

Now, let's try to count the red sets.

If T is a red set, then $T \setminus \{n\}$ is a $(k-1)$-element subset of $\{1, 2, \ldots, n-1\}$.

Let us refer to the $(k-1)$-element subsets of $\{1, 2, \ldots, n-1\}$ as **blue sets**. Thus, if T is a red set, then $T \setminus \{n\}$ is a blue set. Conversely, if U is a blue set, then $U \cup \{n\}$ is a red set. This sets up a one-to-one correspondence between the red sets and the blue sets: We turn red sets into blue sets by removing the element n, and conversely we turn blue sets red by inserting the element n into the set.[3] Hence,

$$\begin{aligned}
(\text{\# of red sets}) &= (\text{\# of blue sets}) \\
&= (\text{\# of } (k-1)\text{-element subsets of } \{1,2,\ldots,n-1\}) \\
&\qquad\qquad (\text{since this is how the blue sets were defined}) \\
&= \binom{n-1}{k-1}
\end{aligned}$$

(again by the statement $P(n-1)$, but now applied to $k-1$ instead of k). Note that we deliberately did not fix k in our induction, so that we were now able to apply $P(n-1)$ to $k-1$ instead of k.

Now, (1) becomes

$$\begin{aligned}
(\text{\# of } k\text{-element subsets of } \{1,2,\ldots,n\}) &= \underbrace{(\text{\# of red sets})}_{=\binom{n-1}{k-1}} + \underbrace{(\text{\# of green sets})}_{=\binom{n-1}{k}} \\
&= \binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}
\end{aligned}$$

by Pascal's recurrence (Theorem 2.5.3 in Lecture 6). But this is precisely the equality that we have to prove. This completes the induction step, and thus Theorem 4.3.3 is proved. $\square$

The above proof can also be used to write an algorithm that lists all the k-element subsets of $\{1,2,\ldots,n\}$. This algorithm is recursive and proceeds as follows:

- If $n = 0$, then:
  - if $k = 0$, then list $\varnothing$ (i.e., the resulting list will consist only of $\varnothing$).
  - otherwise, list nothing.
- Otherwise,
  - list the red sets (by listing all the $(k-1)$-element subsets of $\{1,2,\ldots,n-1\}$, and inserting n into each of them);
  - list the green sets (i.e., the k-element subsets of $\{1,2,\ldots,n-1\}$);

[3] For instance, for $n = 4$ and $k = 2$, this correspondence looks like this:

red set	$\{1,4\}$	$\{2,4\}$	$\{3,4\}$
	$\updownarrow$	$\updownarrow$	$\updownarrow$
blue set	$\{1\}$	$\{2\}$	$\{3\}$

.

  - combine these two lists.

In Python, this algorithm (or one possible implementation of it) looks as follows[4]:

```
def subsets(n, k):
    # listing all subsets of {1, 2, ..., n} that have size k.
    if n == 0:
        if k == 0:
            return [set([])] # set([]) is the empty set
        return [] # empty list
    # Now, the case when n is not 0:
    green_sets = subsets(n-1, k)
    # This is the list of all green sets.
    red_sets = [U.union([n]) for U in subsets(n-1, k-1)]
    # This is the list of all red sets.  We construct it by
    # taking all the (k-1)-element subsets of {1, 2, ..., n-1}
    # (i.e., the blue sets), and inserting n into each of
    # them.
    return red_sets + green_sets
    # In Python, the plus sign can be used to combine two lists.
```

With this code, `subsets(4, 2)` yields

`[{3, 4}, {2, 4}, {1, 4}, {2, 3}, {1, 3}, {1, 2}]`

as an output, and this is indeed a list of all 2-element subsets of $\{1, 2, \ldots, 4\} = \{1, 2, 3, 4\}$.

Theorem 4.3.3 is often called the **combinatorial interpretation of binomial coefficients**, since it reveals that the binomial coefficients $\dbinom{n}{k}$ (at least for $n \in \mathbb{N}$) have a combinatorial meaning (viz., counting k-element subsets of a given n-element set). However, it is just one of many such interpretations, and we will see four others as we progress through this chapter!

4.4. Tuples (aka lists)

4.4.1. Definition and disambiguation

Speaking of lists: What is a finite list? Here is a somewhat awkward definition:

[4]Note that lists are enclosed within brackets in Python: e.g., a list that we call (a, b, c) would be written `[a,b,c]` in Python. Also, Python's notation `set([a,b,c])` corresponds to our $\{a, b, c\}$.

Definition 4.4.1. A **finite list** (aka **tuple**) is a list consisting of finitely many objects. The objects appear in this list in a specified order, and they don't have to be distinct.

A finite list is delimited using parentheses: i.e., the list that contains the objects $a_1, a_2, \ldots, a_n$ in this order is denoted by $(a_1, a_2, \ldots, a_n)$.

"Specified order" means that the list has a well-defined first entry, a well-defined second entry, and so on. Thus, two lists $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_m)$ are considered equal if and only if

- we have $n = m$, and
- we have $a_i = b_i$ for each $i \in \{1, 2, \ldots, n\}$.

For example:

- The lists $(1, 2)$ and $(2, 1)$ are not equal (although the sets $\{1, 2\}$ and $\{2, 1\}$ are equal).
- The lists $(1, 2)$ and $(1, 1, 2)$ are not equal (although the sets $\{1, 2\}$ and $\{1, 1, 2\}$ are equal).
- The lists $(1, 1, 2)$ and $(1, 2, 2)$ are not equal (although the sets $\{1, 1, 2\}$ and $\{1, 2, 2\}$ are equal).

Definition 4.4.2. **(a)** The **length** of a list $(a_1, a_2, \ldots, a_n)$ is defined to be the number n.

(b) A list of length 2 is called a **pair** (or an **ordered pair**).

(c) A list of length 3 is called a **triple**.

(d) A list of length 4 is called a **quadruple**.

(e) A list of length n is called an n**-tuple**.

For example, $(1, 3, 2, 2)$ is a list of length 4 (although it has only 3 **distinct** entries), i.e., a quadruple or a 4-tuple. For another example, $(5, 8)$ is a pair, i.e., a 2-tuple.

Note that there is exactly one list of length 0: the empty list $()$, which contains nothing.

Lists of length 1 consist of just a single entry. For example, (3) is a list containing only the entry 3.

4.4.2. Counting pairs

Now, let us count some pairs:

- How many pairs (a, b) are there with $a, b \in \{1, 2, 3\}$? There are nine:

$$\begin{array}{ccc} (1,1)\,, & (1,2)\,, & (1,3)\,, \\ (2,1)\,, & (2,2)\,, & (2,3)\,, \\ (3,1)\,, & (3,2)\,, & (3,3)\,. \end{array}$$

  The fact that there are nine of them is not surprising given how I've laid them out: They are forming a table with 3 rows and 3 columns, where the row determines the first entry of the pair[5] and the column determines the second entry. Thus, their total number is $3 \cdot 3 = 9$.

- How many pairs (a, b) are there with $a, b \in \{1, 2, 3\}$ and $a < b$? There are three:

$$(1,2)\,, \qquad (1,3)\,, \qquad (2,3)\,.$$

- How many pairs (a, b) are there with $a, b \in \{1, 2, 3\}$ and $a = b$? Again, three:

$$(1,1)\,, \qquad (2,2)\,, \qquad (3,3)\,.$$

- How many pairs (a, b) are there with $a, b \in \{1, 2, 3\}$ and $a > b$? Again, three:

$$(2,1)\,, \qquad (3,1)\,, \qquad (3,2)\,.$$

Let us generalize this:

Proposition 4.4.3. Let $n \in \mathbb{N}$. Then:

(a) The # of pairs (a, b) with $a, b \in \{1, 2, \ldots, n\}$ is n^2.

(b) The # of pairs (a, b) with $a, b \in \{1, 2, \ldots, n\}$ and $a < b$ is $1 + 2 + \cdots + (n-1)$.

(c) The # of pairs (a, b) with $a, b \in \{1, 2, \ldots, n\}$ and $a = b$ is n.

(d) The # of pairs (a, b) with $a, b \in \{1, 2, \ldots, n\}$ and $a > b$ is $1 + 2 + \cdots + (n-1)$.

Informal proof. **(a)** These pairs can be arranged in a table with n rows and n columns, where the rows determine the first entry and the columns determine the second. Here is how this table looks like:

$$\begin{array}{cccc} (1,1)\,, & (1,2)\,, & \ldots, & (1,n)\,, \\ (2,1)\,, & (2,2)\,, & \ldots, & (2,n)\,, \\ \vdots & \vdots & \ddots & \vdots \\ (n,1)\,, & (n,2)\,, & \ldots, & (n,n)\,. \end{array}$$

[5] i.e.:

- The first row contains the pairs that begin with 1.
- The second row contains the pairs that begin with 2.
- The third row contains the pairs that begin with 3.

So there are $n \cdot n = n^2$ of these pairs.

(b) In the table we have just shown, a pair (a,b) satisfies $a < b$ if and only if it is placed above the main diagonal (i.e., the diagonal starting at the northwestern corner and ending at the southeastern corner of the table). Thus, the # of such pairs is the # of cells above the main diagonal in this table. But this # is

$$0 + 1 + 2 + \cdots + (n-1),$$

because there are 0 such cells in the first column, 1 such cell in the second, 2 such cells in the third, and so on. Hence,

$$\begin{aligned}
&(\text{\# of pairs } (a,b) \text{ with } a,b \in \{1,2,\ldots,n\} \text{ and } a < b) \\
&= 0 + 1 + 2 + \cdots + (n-1) \\
&= 1 + 2 + \cdots + (n-1).
\end{aligned}$$

(c) A pair (a,b) with $a = b$ is just a pair of the form (a,a), that is, a single element of $\{1,2,\ldots,n\}$ written twice in succession. Counting such pairs is therefore tantamount to counting single elements of $\{1,2,\ldots,n\}$; but there are clearly n of them.

(d) The pairs (a,b) that satisfy $a > b$ are in one-to-one correspondence with the pairs (a,b) that satisfy $a < b$: Namely, each former pair becomes a latter pair if we swap its two entries, and vice versa. Thus, the # of former pairs equals the # of latter pairs. But we have already found (in part **(b)**) that the # of latter pairs is $1 + 2 + \cdots + (n-1)$. Hence, the # of former pairs is $1 + 2 + \cdots + (n-1)$ as well. $\square$

Proposition 4.4.3 has a nice consequence: For any $n \in \mathbb{N}$, we have

$$\begin{aligned}
n^2 &= (\text{\# of pairs } (a,b) \text{ with } a,b \in \{1,2,\ldots,n\}) \qquad (\text{by Proposition 4.4.3 } \textbf{(a)}) \\
&= \underbrace{(\text{\# of pairs } (a,b) \text{ with } a,b \in \{1,2,\ldots,n\} \text{ and } a < b)}_{\substack{=1+2+\cdots+(n-1) \\ (\text{by Proposition 4.4.3 } \textbf{(b)})}} \\
&\qquad + \underbrace{(\text{\# of pairs } (a,b) \text{ with } a,b \in \{1,2,\ldots,n\} \text{ and } a = b)}_{\substack{=n \\ (\text{by Proposition 4.4.3 } \textbf{(c)})}} \\
&\qquad + \underbrace{(\text{\# of pairs } (a,b) \text{ with } a,b \in \{1,2,\ldots,n\} \text{ and } a > b)}_{\substack{=1+2+\cdots+(n-1) \\ (\text{by Proposition 4.4.3 } \textbf{(d)})}} \\
&\qquad \qquad \left(\begin{array}{c}\text{since each pair } (a,b) \text{ satisfies either } a<b \text{ or } a=b \text{ or } a>b, \\ \text{and never more than one of these three conditions}\end{array}\right) \\
&= \underbrace{(1 + 2 + \cdots + (n-1)) + n}_{=1+2+\cdots+n} + \underbrace{(1 + 2 + \cdots + (n-1))}_{=(1+2+\cdots+n)-n} \\
&= (1 + 2 + \cdots + n) + (1 + 2 + \cdots + n) - n \\
&= 2 \cdot (1 + 2 + \cdots + n) - n.
\end{aligned}$$

Solving this for $1 + 2 + \cdots + n$, we obtain

$$1 + 2 + \cdots + n = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}.$$

Thus, we have recovered the Little Gauss formula (Theorem 1.3.1 in Lecture 2) by counting pairs. This illustrates the fact that counting can be used to prove algebraic identities.

Exercise 1. How many pairs (a, b) are there with $a \in \{1, 2, 3\}$ and $b \in \{1, 2, 3, 4, 5\}$?

Solution. By the same reasoning as in Proposition 4.4.3 **(a)**, there are 15 such pairs, since the pairs can be arranged in a table with 3 rows and 5 columns. □

The same reasoning gives the following more general result:

Theorem 4.4.4. Let $n, m \in \mathbb{N}$. Let A be an n-element set. Let B be an m-element set. Then,

$$(\text{\# of pairs } (a, b) \text{ with } a \in A \text{ and } b \in B) = nm.$$

What about triples?

Theorem 4.4.5. Let $n, m, p \in \mathbb{N}$. Let A be an n-element set. Let B be an m-element set. Let C be a p-element set. Then,

$$(\text{\# of triples } (a, b, c) \text{ with } a \in A \text{ and } b \in B \text{ and } c \in C) = nmp.$$

Informal proof. You can think of these triples as occupying the cells of a 3-dimensional table, but this kind of visualization is tricky (and gets even less reliable when you get to higher dimensions).

A better approach: Re-encode each triple (a, b, c) as a pair $((a, b), c)$ (a pair whose first entry is itself a pair). This is a pair whose first entry comes from the set of all pairs (a, b) with $a \in A$ and $b \in B$, whereas its second entry comes from C. Let U be the set of all pairs (a, b) with $a \in A$ and $b \in B$. Then, this set U is an nm-element set, because

$$\begin{aligned}(\text{\# of elements of } U) &= (\text{\# of pairs } (a, b) \text{ with } a \in A \text{ and } b \in B) \\ &= nm \qquad (\text{by Theorem 4.4.4}).\end{aligned}$$

Now, we have re-encoded each triple (a, b, c) as a pair $((a, b), c)$ with $(a, b) \in U$ and $c \in C$. Thus,

$$\begin{aligned}&(\text{\# of triples } (a, b, c) \text{ with } a \in A \text{ and } b \in B \text{ and } c \in C) \\ &= (\text{\# of pairs } ((a, b), c) \text{ with } (a, b) \in U \text{ and } c \in C) \\ &= (\text{\# of pairs } (u, c) \text{ with } u \in U \text{ and } c \in C) \\ &= (nm)\, p\end{aligned}$$

(by Theorem 4.4.4, since U is an nm-element set while C is a p-element set). In other words,

$$(\text{\# of triples } (a,b,c) \text{ with } a \in A \text{ and } b \in B \text{ and } c \in C) = nmp.$$

This proves Theorem 4.4.5. □

4.4.3. Cartesian products

There is a general notation for sets of pairs:

Definition 4.4.6. Let A and B be two sets.

The set of all pairs (a, b) with $a \in A$ and $b \in B$ is denoted by $A \times B$, and is called the **Cartesian product** (or just **product**) of the sets A and B.

For instance, $\{1,2\} \times \{7,8,9\}$ is the set of all pairs (a, b) with $a \in \{1,2\}$ and $b \in \{7,8,9\}$. Explicitly, it consists of the following six pairs:

$$\begin{array}{lll} (1,7), & (1,8), & (1,9), \\ (2,7), & (2,8), & (2,9). \end{array}$$

A similar notation exists for sets of triples, of quadruples or of k-tuples in general:

Definition 4.4.7. Let $A_1, A_2, \ldots, A_k$ be k sets.

The set of all k-tuples $(a_1, a_2, \ldots, a_k)$ with $a_1 \in A_1$ and $a_2 \in A_2$ and $\cdots$ and $a_k \in A_k$ is denoted by

$$A_1 \times A_2 \times \cdots \times A_k,$$

and is called the **Cartesian product** (or just **product**) of the sets $A_1, A_2, \ldots, A_k$.

For example, the set $\{1,2\} \times \{5\} \times \{2,7,6\}$ consists of all triples (a_1, a_2, a_3) with $a_1 \in \{1,2\}$ and $a_2 \in \{5\}$ and $a_3 \in \{2,7,6\}$. One such pair is $(2,5,2)$; another is $(2,5,6)$. In total, there are $3 \cdot 1 \cdot 2$ such triples (by Theorem 4.4.5).

The word "Cartesian" in "Cartesian product" honors René Descartes, who has observed that a point in the Euclidean plane can be characterized by its two coordinates (i.e., a pair of real numbers), whereas a point in space can be characterized by its three coordinates (i.e., a triple of real numbers). These two observations allow us to think of the plane as the Cartesian product $\mathbb{R} \times \mathbb{R}$, and to think of space as the Cartesian product $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$.

Using the notation $A \times B$, we can restate Theorem 4.4.4 as follows:

Theorem 4.4.8 (product rule for two sets)**.** If A is an n-element set, and B is an m-element set, then $A \times B$ is an nm-element set.

More generally:

Theorem 4.4.9 (product rule for k sets)**.** Let $A_1, A_2, \ldots, A_k$ be k sets. If each A_i is an n_i-element set, then $A_1 \times A_2 \times \cdots \times A_k$ is an $n_1 n_2 \cdots n_k$-element set.

In other words, when you count k-tuples, with each entry coming from a certain set, the total number is the product of the numbers of options for each entry.

You can prove Theorem 4.4.9 by induction on k, using Theorem 4.4.8 and the same "re-encode a tuple as a nested pair" trick that we used in our proof of Theorem 4.4.5. We will later come back to this in more detail.

4.4.4. Counting strictly increasing tuples (informally)

In Proposition 4.4.3 **(b)**, we have seen that for any given $n \in \mathbb{N}$, the # of pairs (a, b) of elements of $\{1, 2, \ldots, n\}$ satisfying $a < b$ is

$$1 + 2 + \cdots + (n-1) = \frac{(n-1)\,n}{2} = \binom{n}{2}.$$

What is the # of triples (a, b, c) of elements of $\{1, 2, \ldots, n\}$ satisfying $a < b < c$?

Such a triple (a, b, c) always determines a 3-element subset $\{a, b, c\}$ of $\{1, 2, \ldots, n\}$ (and yes, this will really be a 3-element subset, because $a < b < c$ entails that a, b, c are distinct). Conversely, any 3-element subset of $\{1, 2, \ldots, n\}$ becomes a triple (a, b, c) with $a < b < c$ if we list its elements in increasing order. Thus, the triples (a, b, c) of elements of $\{1, 2, \ldots, n\}$ satisfying $a < b < c$ are just the 3-element subsets of $\{1, 2, \ldots, n\}$ in disguise.[6] Hence,

$$\begin{aligned}
&(\text{\# of triples } (a,b,c) \text{ of elements of } \{1,2,\ldots,n\} \text{ satisfying } a<b<c) \\
&= (\text{\# of 3-element subsets of } \{1,2,\ldots,n\}) \\
&= \binom{n}{3} \qquad (\text{by Theorem 4.3.3, applied to } S = \{1,2,\ldots,n\} \text{ and } k = 3).
\end{aligned}$$

More generally, for any $k \in \mathbb{N}$, we have

$$\begin{aligned}
&(\text{\# of } k\text{-tuples } (a_1,a_2,\ldots,a_k) \text{ of elements of } \{1,2,\ldots,n\} \text{ satisfying } a_1<a_2<\cdots<a_k) \\
&= \binom{n}{k}
\end{aligned}$$

(by a similar argument: these k-tuples are just the k-element subsets of $\{1, 2, \ldots, n\}$ in disguise). For comparison, if we drop the "$a_1 < a_2 < \cdots < a_k$" requirement,

[6] We are again being informal here. To be more rigorous, we should be speaking of a one-to-one correspondence between the former triples and the latter subsets. But it is not yet the time for this pedantry.

then we have

$$\begin{aligned}
&(\text{\# of } k\text{-tuples } (a_1, a_2, \ldots, a_k) \text{ of elements of } \{1, 2, \ldots, n\}) \\
&= \underbrace{nn \cdots n}_{k \text{ times}} \qquad \text{(by Theorem 4.4.9)} \\
&= n^k.
\end{aligned}$$

Other counting problems don't have answers this simple. For instance, it is not hard to see that

$$\begin{aligned}
&(\text{\# of } k\text{-tuples } (a_1, a_2, \ldots, a_k) \text{ of elements of } \{1, 2, \ldots, n\} \text{ such that } a_1 \text{ is the largest entry}) \\
&= 1^{k-1} + 2^{k-1} + 3^{k-1} + \cdots + n^{k-1},
\end{aligned}$$

but there is no way to express this without a "$\cdots$" or a $\sum$ sign. For each specific k, however, we can simplify this:

$$\begin{aligned}
1^0 + 2^0 + \cdots + n^0 &= \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n; \\
1^1 + 2^1 + \cdots + n^1 &= 1 + 2 + \cdots + n = \frac{n(n+1)}{2}; \\
1^2 + 2^2 + \cdots + n^2 &= \frac{n(n+1)(2n+1)}{6}; \\
1^3 + 2^3 + \cdots + n^3 &= \frac{n^2(n+1)^2}{4}; \\
1^4 + 2^4 + \cdots + n^4 &= \frac{n(2n+1)(n+1)\left(3n + 3n^2 - 1\right)}{30}; \\
&\ldots.
\end{aligned}$$

Such a closed-form expression for $1^m + 2^m + \cdots + n^m$ exists for any specific value of m (see, e.g., [22fco, Lecture 17, Theorem 2.5.3] for how to find it).

Next time, we will learn what it means for a set to have n elements, and what rules we have actually been using in our above informal arguments.

References

[22fco] Darij Grinberg, *Math 222: Enumerative Combinatorics, Fall 2022.* `https://www.cip.ifi.lmu.de/~grinberg/t/22fco/`