Math 221 Winter 2023, Lecture 5: Induction

website: `https://www.cip.ifi.lmu.de/~grinberg/t/23wd`

1. Induction and recursion (cont'd)

1.9. Strong induction (cont'd)

1.9.1. Example: Paying with 3-cent and 5-cent coins

Last time, we discussed the following variant(s) of mathematical induction:

Theorem 1.9.3 (Principle of Strong Induction, standard form)**.** Let b be an integer.

Let $P(n)$ be a mathematical statement defined for each integer $n \geq b$.

Assume the following:

1. "**Base case**": The statement $P(b)$ holds.

2. "**Induction step**": For each integer $n > b$, the implication

$$(P(b) \text{ AND } P(b+1) \text{ AND } P(b+2) \text{ AND } \cdots \text{ AND } P(n-1)) \Longrightarrow P(n)$$

holds.

Then, the statement $P(n)$ holds for every integer $n \geq b$.

Theorem 1.9.4 (Principle of Strong Induction, baseless form)**.** Let b be an integer.

Let $P(n)$ be a mathematical statement defined for each integer $n \geq b$.

Assume the following:

- "**Induction step:**" For each integer $n \geq b$, the implication

$$(P(b) \text{ AND } P(b+1) \text{ AND } P(b+2) \text{ AND } \cdots \text{ AND } P(n-1)) \Longrightarrow P(n)$$

holds.

Then, the statement $P(n)$ holds for every integer $n \geq b$.

Here is another example of how strong induction can be used:

Exercise 1. Assume that you have 3-cent coins and 5-cent coins (each in infinite supply). What denominations can you pay with these coins?

Let's make a table ("yes" means that you can pay it; "no" means that you can't):

0 cents	yes
1 cents	no
2 cents	no
3 cents	yes
4 cents	no
5 cents	yes
6 cents	yes: $2 \cdot 3$
7 cents	no
8 cents	yes: $3 + 5$
9 cents	yes: $3 \cdot 3$
10 cents	yes: $2 \cdot 5$
11 cents	yes: $2 \cdot 3 + 5$
12 cents	yes: $4 \cdot 3$
13 cents	yes: $3 + 2 \cdot 5$
$\cdots$	$\cdots$

.

Experimentally, we seem to observe that any denomination ≥ 8 cents can be paid. Why?

We can notice that if a denomination k (that is, k cents) can be paid, then so can $k + 3$ (just add a 3-cent coin). Thus, because we can pay 8 cents, we can also pay $11, 14, 17, \ldots$ cents. Because we can pay 9 cents, we can also pay $12, 15, 18, \ldots$ cents. Because we can pay 10 cents, we can also pay $13, 16, 19, \ldots$ cents. Together, these three sequences account for all the integers ≥ 8. Thus, any denomination of ≥ 8 cents can be paid.

Let us formalize this argument as an induction proof.

We define $\mathbb{N}$ to be the set of all nonnegative integers:

$$\mathbb{N} = \{0, 1, 2, \ldots\}.$$

Proposition 1.9.7. For any integer $n \geq 8$, we can pay n cents with 3-cent and 5-cent coins. In other words, any integer $n \geq 8$ can be written as $n = 3a + 5b$ with $a, b \in \mathbb{N}$.

Proof. We proceed by strong induction on n:

Base case: For $n = 8$, the claim is true, since $8 = 3 \cdot 1 + 5 \cdot 1$.

Induction step: Fix an integer $n > 8$. Assume that the proposition is already proved for all the integers $8, 9, \ldots, n - 1$. We must prove that it also holds for n.

In other words, we must prove that we can pay n cents with 3-cent and 5-cent coins.

We are in one of the following three cases (since $n > 8$):

Case 1: We have $n = 9$.

Case 2: We have $n = 10$.

Case 3: We have $n \geq 11$.

In Case 1, we are done, since $n = 9 = 3 \cdot 3 + 0 \cdot 5$ (that is, n can be paid with three 3-cent coins).

In Case 2, we are done, since $n = 10 = 0 \cdot 3 + 2 \cdot 5$ (that is, n can be paid with two 5-cent coins).

Now, consider Case 3. In this case, we have $n \geq 11$. Hence, $n - 3 \geq 8$. This shows that $n - 3$ is one of the numbers $8, 9, \ldots, n - 1$.

Thus, we can apply the induction hypothesis to $n - 3$. We conclude that $n - 3$ cents can be paid with 3-cent and 5-cent coins, i.e., we can write $n - 3$ as $n - 3 = 3c + 5d$ with $c, d \in \mathbb{N}$. Using these $c, d \in \mathbb{N}$, we therefore have

$$\begin{aligned} n &= 3 + 3c + 5d \qquad && (\text{since } n - 3 = 3c + 5d) \\ &= 3\left(c + 1\right) + 5d, \end{aligned}$$

which shows that n cents can also be paid with 3-cent and 5-cent coins. This shows that the proposition is true for n, and thus the induction step is complete.

The proposition is thus proved. $\square$

Note that the above proof had one "de-jure base case" (the case $n = 8$) and two "de-facto base cases" (the cases $n = 9$ and $n = 10$, which were formally part of the induction step but had to be treated separately because $n - 3$ would be smaller than 8 in these cases). We could have just as well used the baseless form of strong induction, in which case we would have to treat all three of these cases as "de-facto base cases". This would be a bit more uniform, although this is entirely a matter of taste.

2. Sums and products

2.1. Finite sums

Previously, we have encountered sums such as

$$x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1}$$

(in Lecture 3). Such sums can be tricky to decipher: You need to guess the pattern of the addends to understand what the "$\cdots$" means. There is a notation that makes such sums both shorter and easier to understand. This is the **finite sum notation** (also known as the **sigma notation**). In its simplest form, it is defined as follows:

Definition 2.1.1. Let u and v be two integers. Let $a_u, a_{u+1}, \ldots, a_v$ be some numbers. Then,

$$\sum_{k=u}^{v} a_k$$

is defined to be the sum

$$a_u + a_{u+1} + \cdots + a_v.$$

It is called the **sum of the numbers a_k where k ranges from u to v**. When $v < u$, this sum is called **empty** and defined to be 0.

For example:

$$\sum_{k=5}^{10} k = 5+6+7+8+9+10 = 45;$$

$$\sum_{k=5}^{10} \frac{1}{k} = \frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10} = \frac{2131}{2520};$$

$$\sum_{k=5}^{10} k^k = 5^5+6^6+7^7+8^8+9^9+10^{10};$$

$$\sum_{k=5}^{5} k = 5;$$

$$\sum_{k=5}^{4} k = 0 \qquad \text{(an empty sum)};$$

$$\sum_{k=5}^{3} k = 0 \qquad \text{(an empty sum)};$$

$$\sum_{k=5}^{8} 3 = 3+3+3+3 = 12 \qquad \text{(a sum of four equal terms)};$$

$$\sum_{k=0}^{n-1} q^k = q^0 + q^1 + \cdots + q^{n-1} \qquad \text{for any } n \in \mathbb{N} \text{ and any number } q;$$

$$\begin{aligned}\sum_{k=0}^{n-1} x^k y^{n-1-k} &= y^{n-1} + xy^{n-2} + x^2y^{n-3} + \cdots + x^{n-3}y^2 + x^{n-2}y + x^{n-1} \\ &= x^{n-1} + x^{n-2}y + x^{n-3}y^2 + \cdots + x^2y^{n-3} + xy^{n-2} + y^{n-1} \\ &\qquad \text{for any } n \in \mathbb{N} \text{ and any numbers } x \text{ and } y.\end{aligned}$$

Thus, Theorem 1.6.2 from Lecture 3 is saying that

$$(x-y)\left(\sum_{k=0}^{n-1} x^k y^{n-1-k}\right) = x^n - y^n$$

for any numbers x and y and any $n \in \mathbb{N}$.

The variable k is not set in stone; you can replace it by any other variable (unless this other variable already stands for something else). For example,

$$\sum_{k=u}^{v} a_k = \sum_{i=u}^{v} a_i = \sum_{\text{Ⓢ}=u}^{v} a_{\text{Ⓢ}} = \sum_{\spadesuit=u}^{v} a_{\spadesuit}.$$

Just don't make it $\sum\limits_{u=u}^{v} a_u$.

Here are a couple more examples: For any $n \in \mathbb{N}$, we have

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad \text{(by Theorem 1.3.1 in Lecture 2)};$$

$$\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \cdots + n^2$$
$$= \frac{n(n+1)(2n+1)}{6} \qquad \text{(by Theorem 1.3.2 in Lecture 2)};$$

$$\sum_{k=1}^{n} 1 = \underbrace{1 + 1 + \cdots + 1}_{n \text{ times}} = n \cdot 1 = n;$$

$$\sum_{k=1}^{n} (2k-1) = 1 + 3 + 5 + \cdots + (2n-1)$$
$$= (\text{the sum of the first } n \text{ odd positive integers}).$$

We have not computed this last sum, so let us do this. I will use the following "laws of summation":

- We have
$$\sum_{k=u}^{v} (a_k - b_k) = \sum_{k=u}^{v} a_k - \sum_{k=u}^{v} b_k \tag{1}$$
for any integers u, v and any numbers a_k, b_k. Indeed, if you rewrite this without finite sum notation, it takes the form
$$(a_u - b_u) + (a_{u+1} - b_{u+1}) + \cdots + (a_v - b_v)$$
$$= (a_u + a_{u+1} + \cdots + a_v) - (b_u + b_{u+1} + \cdots + b_v),$$
which is rather clear. (A formal proof can be given by induction on v.)

- We have
$$\sum_{k=u}^{v} \lambda a_k = \lambda \sum_{k=u}^{v} a_k \tag{2}$$
for any integers u, v and any numbers λ, a_k. Indeed, rewritten without the use of finite sum notation, this is just saying that
$$\lambda a_u + \lambda a_{u+1} + \cdots + \lambda a_v = \lambda (a_u + a_{u+1} + \cdots + a_v),$$
which is again clear (and can be proved by induction on v).

Rules like this are dime a dozen, and you should be able to come up with them on the spot when you need them. (See [Grinbe15, §1.4.2] for these and several others.)

Let us now compute our sum:

$$\begin{aligned}
\sum_{k=1}^{n} (2k-1) &= \sum_{k=1}^{n} 2k - \sum_{k=1}^{n} 1 && \text{(by (1))} \\
&= 2 \underbrace{\sum_{k=1}^{n} k}_{=\frac{n(n+1)}{2}} - \underbrace{\sum_{k=1}^{n} 1}_{=n} && \text{(by (2))} \\
&= 2 \cdot \frac{n(n+1)}{2} - n = n(n+1) - n = n^2.
\end{aligned}$$

As another illustration of our method, we can rewrite Gauss's proof of the equality

$$\sum_{k=1}^{n} k = 1 + 2 + \cdots + n = \frac{n(n+1)}{2} \qquad (3)$$

(Theorem 1.3.2 in Lecture 2) using finite sum notation. We will need two new rules this time:

- We have
$$\sum_{k=u}^{v} a_k + \sum_{k=u}^{v} b_k = \sum_{k=u}^{v} (a_k + b_k) \qquad (4)$$
for any integers u, v and any numbers a_k, b_k. Indeed, if you rewrite this without finite sum notation, it takes the form
$$\begin{aligned}
&(a_u + a_{u+1} + \cdots + a_v) + (b_u + b_{u+1} + \cdots + b_v) \\
&= (a_u + b_u) + (a_{u+1} + b_{u+1}) + \cdots + (a_v + b_v).
\end{aligned}$$

- We have
$$\sum_{k=u}^{v} a_k = \sum_{k=u}^{v} a_{u+v-k} \qquad (5)$$
for any integers u, v and any numbers a_k. This is called "substituting $u + v - k$ for k in the sum" or just "turning the sum upside-down", as it amounts to reversing the order of the addends; restated without finite sum notation, this is just saying that
$$a_u + a_{u+1} + \cdots + a_v = a_v + a_{v-1} + \cdots + a_u,$$
which is saying that a sum of a bunch of numbers does not change if we add its addends together in reverse order.

- For any integers $u \leq v$ and any number λ, we have

$$\sum_{k=u}^{v} \lambda = (v - u + 1)\,\lambda. \tag{6}$$

  (This is just saying that a sum of $v - u + 1$ many equal addends λ is $(v - u + 1)\,\lambda$. Note that the sum on the left hand side has $v - u + 1$ addends, because there are $v - u + 1$ numbers in the set $\{u, u + 1, \ldots, v\}$.)

Now, Gauss's proof of (3) takes the following shape:

$$\begin{aligned}
2\sum_{k=1}^{n} k &= \sum_{k=1}^{n} k + \sum_{k=1}^{n} k \\
&= \sum_{k=1}^{n} k + \sum_{k=1}^{n} (n + 1 - k) \\
&\qquad\qquad \left(\begin{array}{c}\text{here, we substituted } n+1-k \text{ for } k \text{ in the second} \\ \text{sum (i.e., rewrote it using (5))}\end{array}\right) \\
&= \sum_{k=1}^{n} \underbrace{(k + (n + 1 - k))}_{=n+1} \qquad \text{(by (4))} \\
&= \sum_{k=1}^{n} (n + 1) = n \cdot (n + 1) \qquad \text{(by (6))}.
\end{aligned}$$

Dividing both sides by 2, we recover (3) again.

We have found closed-form expressions (i.e., expressions without $\sum$ signs or "$\cdots$"s) for several sums. Not every sum has a closed-form expression. For instance, there is no closed form for

$$\sum_{k=1}^{n} \frac{1}{k} \qquad \text{or for} \qquad \sum_{k=1}^{n} k^k.$$

Some more terminology:

The notation $\sum\limits_{k=u}^{v} a_k$ is called **sigma notation** or **finite sum notation**. The symbol $\sum$ itself is called the **summation sign**. The numbers u and v are called the **lower limit** and the **upper limit** of the summation[1]. The variable k is called the **summation index** or the **running index**, and is said to **range** (or **run**) from u to v. The numbers a_k are called the **addends** of the finite sum.

[1]This use of the word "limit" is totally unrelated to the way this word is used in analysis/calculus.

There are many similarities between finite sums $\sum\limits_{k=u}^{v} a_k$ and integrals $\int_u^v f(x)\, dx$, but the analogy should not be taken too far (e.g., an integral $\int_u^u f(x)\, dx$ whose upper and lower limit are equal will always be 0, but an "analogous" finite sum $\sum\limits_{k=u}^{u} a_k$ will be a_u).

We note two more rules for finite sums:

- The "splitting-off rule": For any integers $u \leq v$ and any numbers $a_u, a_{u+1}, \ldots, a_v$, we have
$$\sum_{k=u}^{v} a_k = \sum_{k=u}^{v-1} a_k + a_v = a_u + \sum_{k=u+1}^{v} a_k.$$
This is just saying that
$$a_u + a_{u+1} + \cdots + a_v = (a_u + a_{u+1} + \cdots + a_{v-1}) + a_v = a_u + (a_{u+1} + a_{u+2} + \cdots + a_v).$$
This rule allows us to split the first or the last addend out of a finite sum. This is important for proofs by induction.

- More generally, any finite sum $\sum\limits_{k=u}^{v} a_k$ can be split at any point: We have
$$\sum_{k=u}^{v} a_k = \sum_{k=u}^{w} a_k + \sum_{k=w+1}^{v} a_k$$
for any integers $u \leq w \leq v$ and any numbers a_k. This is just saying that
$$a_u + a_{u+1} + \cdots + a_v = (a_u + a_{u+1} + \cdots + a_w) + (a_{w+1} + a_{w+2} + \cdots + a_v).$$
(Strictly speaking, this is true not just for $u \leq w \leq v$ but more generally for $u - 1 \leq w \leq v$.)

Finite sum notation, in the form defined above, is helpful when the summation index is running over an integer interval (i.e., a set of consecutive integers). For more general situations, there is a more general version of finite sum notation, e.g.:
$$\sum_{k \in \{1,2,\ldots,n\} \text{ is even}} k = 2 + 4 + 6 + \cdots + m,$$
where m is the largest even element of $\{1, 2, \ldots, n\}$. We won't use it much, but it is fairly self-explanatory; essentially, the writing under the summation sign explains what k's the sum is ranging over. See [Grinbe15, §1.4.1] for a more precise explanation.

2.2. Finite products

Finite products are analogous to finite sums, just using multiplication instead of addition:

> **Definition 2.2.1.** Let u and v be two integers. Let $a_u, a_{u+1}, \ldots, a_v$ be some numbers. Then,
> $$\prod_{k=u}^{v} a_k$$
> is defined to be the product
> $$a_u a_{u+1} \cdots a_v.$$
> It is called the **product of the numbers a_k where k ranges from u to v**. When $v < u$, this product is called **empty** and defined to be 1.

For example:

$$\prod_{k=5}^{10} k = 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = 151\,200;$$

$$\prod_{k=1}^{5} \frac{1}{k} = \frac{1}{1} \cdot \frac{1}{2} \cdot \frac{1}{3} \cdot \frac{1}{4} \cdot \frac{1}{5} = \frac{1}{120};$$

$$\prod_{k=1}^{n} a = \underbrace{aa \cdots a}_{n \text{ times}} = a^n \qquad \text{for any fixed number } a;$$

$$\begin{aligned}
\prod_{k=1}^{n} a^k &= a^1 a^2 \cdots a^n \\
&= a^{1+2+\cdots+n} \qquad \left(\begin{array}{c}\text{by one of the laws of exponents:} \\ \text{namely, the law } a^{i_1} a^{i_2} \cdots a^{i_n} = a^{i_1+i_2+\cdots+i_n}\end{array}\right) \\
&= a^{n(n+1)/2} \qquad \text{for any fixed number } a \text{ and any } n \in \mathbb{N}.
\end{aligned}$$

In a finite product $\prod\limits_{k=u}^{v} a_k$, the k is called the **product index** or the **running index**, and the symbol $\prod$ is called the **product sign**. The numbers a_k are called the **factors** of the product. Other terminology is analogous to the case of a finite sum (e.g., lower limit, upper limit). Almost all rules for finite sums have analogues for finite products. Let me only state the analogue of the "splitting-off rule":

- The "splitting-off rule" for products: For any integers $u \leq v$ and any numbers $a_u, a_{u+1}, \ldots, a_v$, we have
$$\prod_{k=u}^{v} a_k = \left(\prod_{k=u}^{v-1} a_k\right) a_v = a_u \prod_{k=u+1}^{v} a_k.$$

This is just saying that

$$a_u a_{u+1} \cdots a_v = \left(a_u a_{u+1} \cdots a_{v-1}\right) a_v = a_u \left(a_{u+1} a_{u+2} \cdots a_v\right).$$

This rule allows us to split the first or the last factor out of a finite product. This is important for proofs by induction.

2.3. Factorials

Now, we define a sequence of integers that appears all over mathematics. Recall that $\mathbb{N} = \{0, 1, 2, \ldots\}$.

Definition 2.3.1. For any $n \in \mathbb{N}$, we define the positive integer $n!$ (called the **factorial** of n, and often pronounced "n **factorial**") by

$$n! = \prod_{k=1}^{n} k = 1 \cdot 2 \cdot \cdots \cdot n.$$

For example,

$$\begin{aligned}
0! &= (\text{empty product}) = 1; \\
1! &= 1 = 1; \\
2! &= 1 \cdot 2 = 2; \\
3! &= 1 \cdot 2 \cdot 3 = 6; \\
4! &= 1 \cdot 2 \cdot 3 \cdot 4 = 24; \\
5! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120; \\
6! &= 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 = 720; \\
7! &= 5\,040; \\
8! &= 40\,320; \\
9! &= 362\,880; \\
10! &= 3\,628\,800.
\end{aligned}$$

Note the following:

Proposition 2.3.2 (recursion of the factorials)**.** For any positive integer n, we have

$$n! = (n-1)! \cdot n.$$

Proof. Let n be a positive integer. Then,

$$n! = 1 \cdot 2 \cdot \cdots \cdot n = \underbrace{(1 \cdot 2 \cdot \cdots \cdot (n-1))}_{=(n-1)!} \cdot n = (n-1)! \cdot n.$$

$\square$

2.4. Binomial coefficients: Definition

We shall now define one of the most important families of numbers in mathematics:

Definition 2.4.1. Let n and k be any numbers. Then, we define a number $\dbinom{n}{k}$ as follows:

- If $k \in \mathbb{N}$, then we set

$$\binom{n}{k} := \frac{n\left(n-1\right)\left(n-2\right)\cdots\left(n-k+1\right)}{k!}$$

(where the numerator is the product of k consecutive integers, the largest of which is n; you can also write it as $\prod\limits_{i=0}^{k-1}\left(n-i\right)$).

- If $k \notin \mathbb{N}$, then we set

$$\binom{n}{k} := 0.$$

The number $\dbinom{n}{k}$ is called "n **choose** k", and is known as the **binomial coefficient** of n and k. Do not mistake the notation $\dbinom{n}{k}$ for a vector $\left(\begin{array}{c} n \\ k \end{array}\right)$.

Example 2.4.2. For any number n, we have

$$\begin{aligned}
\binom{n}{3} &= \frac{n\left(n-1\right)\left(n-2\right)}{3!} = \frac{n\left(n-1\right)\left(n-2\right)}{6}; \\
\binom{n}{2} &= \frac{n\left(n-1\right)}{2!} = \frac{n\left(n-1\right)}{2}; \\
\binom{n}{1} &= \frac{n}{1!} = n; \\
\binom{n}{0} &= \frac{\text{(empty product)}}{0!} = \frac{1}{1} = 1; \\
\binom{n}{2.5} &= 0 \qquad \left(\text{since } 2.5 \notin \mathbb{N}\right); \\
\binom{n}{-1} &= 0 \qquad \left(\text{since } -1 \notin \mathbb{N}\right).
\end{aligned}$$

For any $k \in \mathbb{N}$, we have

$$\binom{0}{k} = \frac{0\,(0-1)\,(0-2)\cdots(0-k+1)}{k!} = \begin{cases} 1, & \text{if } k = 0; \\ 0, & \text{if } k \neq 0; \end{cases}$$

$$\binom{-1}{k} = \frac{(-1)\,(-2)\,(-3)\cdots(-k)}{k!} = (-1)^k \cdot \underbrace{\frac{1 \cdot 2 \cdot \cdots \cdot k}{k!}}_{=1} = (-1)^k.$$

Let us tabulate the values of $\binom{n}{k}$ for nonnegative integers n and k:

	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$	$k=5$	$k=6$
$n=0$	1	0	0	0	0	0	0
$n=1$	1	1	0	0	0	0	0
$n=2$	1	2	1	0	0	0	0
$n=3$	1	3	3	1	0	0	0
$n=4$	1	4	6	4	1	0	0
$n=5$	1	5	10	10	5	1	0
$n=6$	1	6	15	20	15	6	1

.

What patterns can we spot in this table? (We are ignoring negative and non-integer n's for now.)

Proposition 2.4.3. Let $n \in \mathbb{N}$ and $k > n$. Then, $\binom{n}{k} = 0$.

Proof. If $k \notin \mathbb{N}$, then this is clear by definition. Otherwise, again by definition, we have

$$\binom{n}{k} = \frac{n\,(n-1)\,(n-2)\cdots(n-k+1)}{k!} = \frac{0}{k!}$$

(since the product $n\,(n-1)\,(n-2)\cdots(n-k+1)$ has a factor of $n-n=0$, and thus is 0). For example, for $n=3$ and $k=6$, we have

$$\binom{3}{6} = \frac{3 \cdot 2 \cdot 1 \cdot 0 \cdot (-1) \cdot (-2)}{6!} = \frac{0}{6!} = 0.$$

□

> **Remark 2.4.4.** Note that Proposition 2.4.3 would **not** hold without the $n \in \mathbb{N}$ assumption. For example,
>
> $$\binom{1.5}{3} = \frac{1.5 \cdot 0.5 \cdot (-0.5)}{3!} \neq 0 \qquad \text{even though } 3 > 1.5.$$
>
> The product in the numerator is not 0, since it "misses" the 0 factor.

Proposition 2.4.3 explains why our above table of $\binom{n}{k}$ has so many zeroes in it. More precisely, it tells us that all entries above the main diagonal of the table are zeroes (no matter how many more rows and columns we add). Thus, we can redraw our table as a triangular table (and fill in a few more rows while at that):

$$\begin{array}{rl|c}
 & & k{=}0 \swarrow \\
n=0 & \rightarrow & 1 \quad k{=}1 \swarrow \\
n=1 & \rightarrow & 1 \quad 1 \quad k{=}2 \swarrow \\
n=2 & \rightarrow & 1 \quad 2 \quad 1 \quad k{=}3 \swarrow \\
n=3 & \rightarrow & 1 \quad 3 \quad 3 \quad 1 \quad k{=}4 \swarrow \\
n=4 & \rightarrow & 1 \quad 4 \quad 6 \quad 4 \quad 1 \quad k{=}5 \swarrow \\
n=5 & \rightarrow & 1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \quad k{=}6 \swarrow \\
n=6 & \rightarrow & 1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \quad k{=}7 \swarrow \\
n=7 & \rightarrow & 1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
n=8 & \rightarrow & 1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1
\end{array}$$

This table is known as **Pascal's triangle**, and has a variety of wonderful properties. Here are just a few:

> **Proposition 2.4.5** (Pascal's identity, aka the recurrence of the binomial coefficients)**.** For any numbers n and k, we have
>
> $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

> **Proposition 2.4.6** (symmetry of binomial coefficients)**.** For any $n \in \mathbb{N}$ and any k, we have $\binom{n}{k} = \binom{n}{n-k}$.

Proposition 2.4.7. We have $\dbinom{n}{n} = 1$ for each $n \in \mathbb{N}$.

Proposition 2.4.8 (integrality of binomial coefficients)**.** For any $n \in \mathbb{Z}$ and any k, we have $\dbinom{n}{k} \in \mathbb{Z}$.

Next time, we will prove these four propositions and more.

References

[Grinbe15] Darij Grinberg, *Notes on the combinatorial fundamentals of algebra*, 15 September 2022, arXiv:2008.09862v3.