Math 221 Winter 2023, Lecture 3: Induction

website: `https://www.cip.ifi.lmu.de/~grinberg/t/23wd`

1. Induction and recursion (cont'd)

1.6. Some more examples of induction

Let us see some more examples of proofs by induction.

Theorem 1.6.1. For any integer $n \geq 0$, we have

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

Proof. We induct on n.

Base case: For $n = 0$, the equality $2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$ is true, because the LHS[1] is an empty sum and thus equals 0, whereas the RHS is $2^0 - 1 = 1 - 1 = 0$.

Induction step: Let n be an integer ≥ 0. Assume that Theorem 1.6.1 holds for n, i.e., that we have

$$2^0 + 2^1 + 2^2 + \cdots + 2^{n-1} = 2^n - 1.$$

We must prove that Theorem 1.6.1 holds for $n + 1$ as well, i.e., that we have

$$2^0 + 2^1 + 2^2 + \cdots + 2^{(n+1)-1} = 2^{n+1} - 1.$$

However,

$$\begin{aligned} & 2^0 + 2^1 + 2^2 + \cdots + 2^{(n+1)-1} \\ & = 2^0 + 2^1 + 2^2 + \cdots + 2^n \\ & = \underbrace{\left(2^0 + 2^1 + 2^2 + \cdots + 2^{n-1}\right)}_{\substack{=2^n - 1 \\ \text{(by the induction hypothesis)}}} + 2^n \\ & = 2^n - 1 + 2^n = \underbrace{2 \cdot 2^n}_{=2^{n+1}} - 1 = 2^{n+1} - 1, \end{aligned}$$

which is precisely what we want: This shows that Theorem 1.6.1 holds for $n + 1$. Thus, our induction step is complete, and Theorem 1.6.1 is proved. □

Theorem 1.6.1 can be generalized:

[1] "LHS" means "left-hand side". Likewise, "RHS" means "right-hand side".

> **Theorem 1.6.2.** Let x and y be any two numbers. Then, for any integer $n \geq 0$, we have
>
> $$(x-y)\left(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1}\right)=x^n-y^n.$$
>
> Here, the big sum in the parentheses is the sum of all products x^iy^j where i and j are nonnegative integers with $i+j=n-1$.

Before we prove this, let us give some examples for what this theorem actually says:

- For $n=2$, Theorem 1.6.2 says that

  $$(x-y)(x+y)=x^2-y^2.$$

- For $n=3$, Theorem 1.6.2 says that

  $$(x-y)\left(x^2+xy+y^2\right)=x^3-y^3.$$

- For $n=4$, Theorem 1.6.2 says that

  $$(x-y)\left(x^3+x^2y+xy^2+y^3\right)=x^4-y^4.$$

- For $x=2$ and $y=1$, Theorem 1.6.2 says that

  $$(2-1)\left(2^{n-1}+2^{n-2}1+2^{n-3}1^2+\cdots+2^21^{n-3}+2\cdot 1^{n-2}+1^{n-1}\right)=2^n-1^n.$$

  Since any power of 1 is 1 (and since the $2-1$ factor also equals 1), this simplifies to

  $$2^{n-1}+2^{n-2}+2^{n-3}+\cdots+2^2+2+1=2^n-1,$$

  which is precisely Theorem 1.6.1. Thus, Theorem 1.6.2 generalizes Theorem 1.6.1.

Let us now prove Theorem 1.6.2:

Proof of Theorem 1.6.2. We induct on n.

Base case: For $n=0$, the claim

$$(x-y)\left(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1}\right)=x^n-y^n$$

is true, since the LHS is 0 (because the second factor is an empty sum), while the RHS is $x^0-y^0=1-1=0$ as well.

Induction step: Let $n \geq 0$ be an integer. Assume that Theorem 1.6.2 is true for n. That is, assume that

$$(x-y)\left(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1}\right)=x^n-y^n.$$

We must prove that Theorem 1.6.2 is also true for $n+1$. That is, we must prove that

$$(x-y)\left(x^{n}+x^{n-1}y+x^{n-2}y^2+\cdots+x^3y^{n-3}+x^2y^{n-2}+xy^{n-1}+y^{n}\right)=x^{n+1}-y^{n+1}.$$

We begin by extracting the y^n addend from the long sum in the second pair of parentheses in this equation. We thus obtain

$$\begin{aligned}
&(x-y)\left(x^{n}+x^{n-1}y+x^{n-2}y^2+\cdots+x^3y^{n-3}+x^2y^{n-2}+xy^{n-1}+y^{n}\right)\\
&=(x-y)\underbrace{\left(x^{n}+x^{n-1}y+x^{n-2}y^2+\cdots+x^3y^{n-3}+x^2y^{n-2}+xy^{n-1}\right)}_{\substack{=\left(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1}\right)x \\ \text{(here, we have factored out an } x \text{ from the sum)}}}+(x-y)\,y^n\\
&=\underbrace{(x-y)\left(x^{n-1}+x^{n-2}y+x^{n-3}y^2+\cdots+x^2y^{n-3}+xy^{n-2}+y^{n-1}\right)}_{\substack{=x^n-y^n \\ \text{(by the induction hypothesis)}}}x+(x-y)\,y^n\\
&=(x^n-y^n)\,x+(x-y)\,y^n=x^{n+1}-xy^n+xy^n-y^{n+1}=x^{n+1}-y^{n+1}.
\end{aligned}$$

This means precisely that Theorem 1.6.2 is also true for $n+1$. Thus, the induction step is complete, and the theorem is proved. □

Another useful particular case of Theorem 1.6.2 is the following equality:[2]

Corollary 1.6.3. Let q be a number distinct from 1. Let $n \geq 0$ be an integer. Then,

$$q^0+q^1+q^2+\cdots+q^{n-1}=\frac{q^n-1}{q-1}.$$

Proof. Apply Theorem 1.6.2 to $x=q$ and $y=1$. We obtain

$$(q-1)\left(q^{n-1}+q^{n-2}1+q^{n-3}1^2+\cdots+q^21^{n-3}+q\cdot 1^{n-2}+1^{n-1}\right)=q^n-1^n.$$

Simplifying this, we obtain

$$(q-1)\left(q^{n-1}+q^{n-2}+q^{n-3}+\cdots+q^2+q+1\right)=q^n-1.$$

Thus,

$$q^{n-1}+q^{n-2}+q^{n-3}+\cdots+q^2+q+1=\frac{q^n-1}{q-1}.$$

[2] A "corollary" means a theorem that follows easily from another theorem.

In other words,

$$q^0 + q^1 + q^2 + \cdots + q^{n-1} = \frac{q^n - 1}{q - 1}$$

(since the sum on the left hand side can be rearranged in any order). This proves Corollary 1.6.3. □

1.7. How not to use induction

Induction proofs can be slippery:

Theorem 1.7.1 (Fake theorem)**.** In any set of $n \geq 1$ horses, all the horses are the same color.

Proof. We induct on n.

Base case: This is clearly true for $n = 1$, since a single horse always has the same color as itself.

Induction step: Let $n \geq 1$ be an integer. We assume that the theorem holds for n, i.e., that any n horses are the same color.

We must prove that it also holds for $n + 1$, i.e., that any $n + 1$ horses are the same color.

So let $H_1, H_2, \ldots, H_{n+1}$ be $n + 1$ horses.

By our induction hypothesis, the first n horses $H_1, H_2, \ldots, H_n$ are the same color.

Again by our induction hypothesis, the last n horses $H_2, H_3, \ldots, H_{n+1}$ are the same color.

Now, consider the first horse H_1 and the last horse H_{n+1}. They both have the same color as the "middle horses" $H_2, H_3, \ldots, H_n$ (according to the preceding two paragraphs). Thus, all the $n + 1$ horses have the same color, right?

When a claim is as obviously wrong as this one, there is an easy way to find the mistake in the proof: You just look at some example in which the claim is wrong, and you trace the proof on this example. The first time you see a wrong conclusion, that's where the error probably is.

Theorem 1.7.1 is wrong for $n = 2$ already, i.e., for two horses. So let us see where the induction step goes wrong when $n = 1$ (that is, going from 1 horse to 2 horses). In this induction step, we claim that H_1 and $H_{n+1} = H_2$ both have the same color as the "middle horses" $H_2, H_3, \ldots, H_1$. But there are no "middle horses", so it makes no sense to have the same color as these "middle horses". So the argument doesn't work.

Thus, our mistake was to implicitly treat the "middle horses" as if they existed. They do exist for any $n > 1$, but not for $n = 1$, and thus our induction step breaks down for $n = 1$. □

Note how one little mistake has brought down the entire proof! For an induction proof to work, the induction step needs to work for all n; that is, we need the implication $P(n) \Longrightarrow P(n+1)$ to hold for every n. If even one of these implications breaks down, the whole chain is disconnected, and all the statements $P(n)$ "to the right of" this breaking point are no longer guaranteed to hold. For example, if we have a statement $P(n)$ for each $n \geq 0$, and we have proved the base case $P(0)$ and the implication $P(n) \Longrightarrow P(n+1)$ for all $n \neq 4$, then we can conclude that $P(0)$, $P(1)$, $P(2)$, $P(3)$ and $P(4)$ hold, but we cannot guarantee that any of $P(5)$, $P(6)$, $P(7)$, ... hold. As so often, a chain is only as strong as its weakest link.

1.8. More on the Fibonacci numbers

Recall the Fibonacci sequence, which we defined in Lecture 2:

Definition 1.8.1. The **Fibonacci sequence** is the sequence $(f_0, f_1, f_2, \ldots)$ of nonnegative integers defined recursively by setting

$$f_0 = 0, \qquad f_1 = 1, \qquad \text{and}$$
$$f_n = f_{n-1} + f_{n-2} \qquad \text{for each } n \geq 2.$$

The entries of the Fibonacci sequence are called the **Fibonacci numbers**. Here are the first few:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13
f_n	0	1	1	2	3	5	8	13	21	34	55	89	144	233

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Back in Lecture 2, we proved:

Theorem 1.8.2. For any integer $n \geq 0$, we have

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

Let us now prove two deeper properties of the Fibonacci sequence.

Theorem 1.8.3 (addition theorem for Fibonacci numbers)**.** We have

$$f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1} \qquad \text{for all integers } n, m \geq 0.$$

Proof. Can you induct on two variables at the same time? Not directly (although you can induct on n and then induct on m in the induction step, so that you have one induction proof inside another). Fortunately, we don't need to do this here. It suffices to induct on one of the variables.

To be specific, let us induct on n. To that purpose, for every integer $n \geq 0$, we define the statement $P(n)$ to say

$$\text{"for all integers } m \geq 0\text{, we have } f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}\text{".}$$

(Don't forget the "for all integers $m \geq 0$" part! The statement $P(n)$ is not just a single equality $f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$ for some specific value of m, but rather combines infinitely many such equalities, one for each integer $m \geq 0$. If we fixed a value of m and defined $P(n)$ to be just the single equality $f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$, then the induction proof below would not work, because we are going to apply the induction hypothesis to a different m than we start with.)

We shall now prove this statement $P(n)$ for all $n \geq 0$ by induction on n.

Base case: We must prove $P(0)$. In other words, we must prove that

$$\text{"for all integers } m \geq 0\text{, we have } f_{0+m+1} = f_0 f_m + f_{0+1} f_{m+1}\text{".}$$

This is easy to show: For all integers $m \geq 0$, we have $f_{0+m+1} = f_{m+1}$ and $\underbrace{f_0}_{=0} f_m + \underbrace{f_{0+1}}_{=f_1=1} f_{m+1} = 0 f_m + 1 f_{m+1} = f_{m+1}$, so the two sides are equal.

Induction step: Let $n \geq 0$ be an integer. We assume that $P(n)$ holds. We must show that $P(n+1)$ holds.

Our induction hypothesis says that $P(n)$ holds, i.e., that

$$\text{"for all integers } m \geq 0\text{, we have } f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}\text{" holds.}$$

We must prove that $P(n+1)$ holds, i.e., that

$$\text{"for all integers } m \geq 0\text{, we have } f_{n+1+m+1} = f_{n+1} f_m + f_{n+1+1} f_{m+1}\text{" holds.}$$

To prove this, we let $m \geq 0$ be an integer. Then,

$$\begin{aligned}
& f_{n+1} f_m + \underbrace{f_{n+1+1}}_{\substack{=f_{n+2} \\ =f_{n+1}+f_n \\ \text{(by the recursive} \\ \text{definition of the} \\ \text{Fibonacci numbers)}}} f_{m+1} \\
&= f_{n+1} f_m + \left(f_{n+1} + f_n\right) f_{m+1} \\
&= f_{n+1} f_m + f_{n+1} f_{m+1} + f_n f_{m+1} \\
&= f_{n+1} \underbrace{\left(f_m + f_{m+1}\right)}_{\substack{=f_{m+1}+f_m \\ =f_{m+2} \\ \text{(by the recursive} \\ \text{definition of the} \\ \text{Fibonacci numbers)}}} + f_n f_{m+1} \\
&= f_{n+1} f_{m+2} + f_n f_{m+1} = f_n f_{m+1} + f_{n+1} f_{m+2}. \qquad (1)
\end{aligned}$$

Now, recall that the induction hypothesis says that $P(n)$ holds, i.e., that

"for all integers $m \geq 0$, we have $f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$" holds.

Note that the m in this statement is a bound variable, i.e., it has nothing to do with the m that we have fixed; it just happens to have the same name. Thus, we are free to apply our induction hypothesis $P(n)$ not to the current m, but to any other m as well. In particular, we can apply it to $m+1$ instead of m. Thus, we obtain

$$f_{n+m+1+1} = f_n f_{m+1} + f_{n+1} f_{m+1+1}.$$

[3] This can be trivially simplified to

$$f_{n+m+2} = f_n f_{m+1} + f_{n+1} f_{m+2}.$$

This equality has the same right hand side as (1). Thus, the left hand sides of the two equalities must be equal as well. In other words, we must have

$$f_{n+m+2} = f_{n+1} f_m + f_{n+1+1} f_{m+1}.$$

Since $n+m+2 = n+1+m+1$, we can rewrite this as

$$f_{n+1+m+1} = f_{n+1} f_m + f_{n+1+1} f_{m+1}.$$

Thus, we have proved that for all integers $m \geq 0$, we have $f_{n+1+m+1} = f_{n+1} f_m + f_{n+1+1} f_{m+1}$. In other words, we have proved that $P(n+1)$ holds.

So the induction step is complete, and Theorem 1.8.3 is proved. □

Our next theorem involves divisibility of integers. We will study this in more detail in future lectures (it is the fundamental concept of number theory), but for now let me give its definition:

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[3]Let me explain this again in a slightly clearer (if longer) way.

Our induction hypothesis tells us that

"for all integers $m \geq 0$, we have $f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$" holds.

We can rename the variable m as p in this statement (since it is just a bound variable). Thus, we obtain that

"for all integers $p \geq 0$, we have $f_{n+p+1} = f_n f_p + f_{n+1} f_{p+1}$" holds.

Now, applying this latter statement to $p = m+1$ (where m is the m that we fixed), we obtain

$$f_{n+m+1+1} = f_n f_{m+1} + f_{n+1} f_{m+1+1}.$$

Definition 1.8.4. Let a and b be two integers. We say that a **divides** b (and we write $a \mid b$) if there exists an integer c such that $b = ac$. Equivalently, we say that b is **divisible by** a in this case.

For example, we have $2 \mid 4$ and $3 \mid 12$ and $10 \mid 30$ and $0 \mid 0$ and $5 \mid 0$. But we don't have $2 \mid 3$ or $0 \mid 1$. The integer 0 is divisible by every integer, but only divides itself.

Now we can state a divisibility property of Fibonacci numbers:

Theorem 1.8.5. If $a, b \geq 0$ are two integers that satisfy $a \mid b$, then $f_a \mid f_b$.

In other words, in our above table of Fibonacci numbers, if some entry of the first row divides some other entry of the first row, then the same holds for the corresponding entries of the second row. For example, $6 \mid 12$ implies $f_6 \mid f_{12}$ (which is saying that $8 \mid 144$).

Proof of Theorem 1.8.5. It is reasonable to try induction. However, inducting on a does not lead anywhere: The base case is easy, but in the induction step it is completely unclear how to reach the goal, since the condition $a \mid b$ in the induction hypothesis usually has nothing to do with the condition $a + 1 \mid b$ in the induction goal.

Similar problems appear if you try to induct on b. So neither of the two variables in the theorem is suitable for being inducted on.

What can we do? Give up on induction?

Not so fast. One thing we haven't tried is to introduce a new variable and then induct on that new variable.

To do so, we observe that two integers $a, b \geq 0$ satisfy $a \mid b$ if and only if there exists an integer c such that $b = ac$ (by the definition of "divides"). Moreover, if this integer c exists, then it can be chosen to be ≥ 0 (this is automatic when $b \neq 0$, because $c = \frac{b}{a} > 0$ in this case; but otherwise we can achieve this by simply choosing $c = 0$). Thus, two integers $a, b \geq 0$ satisfy $a \mid b$ if and only if there exists an integer $c \geq 0$ such that $b = ac$.

Hence, a pair of integers $a, b \geq 0$ satisfying $a \mid b$ is nothing but a pair of the form a, ac where $a, c \geq 0$ are integers. This allows us to restate Theorem 1.8.5 as follows:

Restated theorem: "For any integers $a, c \geq 0$, we have $f_a \mid f_{ac}$."

Now, we shall prove this restated theorem by induction on c. In other words, for each $c \geq 0$, we shall prove the statement

$$P(c) := (\text{"for any integer } a \geq 0 \text{, we have } f_a \mid f_{ac}\text{"}).$$

Base case: We must prove $P(0)$. In other words, we must prove that

$$\text{"for any integer } a \geq 0 \text{, we have } f_a \mid f_{a \cdot 0}\text{"}.$$

But this is easy, because for any integer $a \geq 0$, we have $f_{a \cdot 0} = f_0 = 0$, which is divisible by any integer (thus in particular by f_a).

Induction step: Let $c \geq 0$ be an integer. We assume that $P(c)$ holds, i.e., that

"for any integer $a \geq 0$, we have $f_a \mid f_{ac}$" holds.

We must prove that $P(c+1)$ holds, i.e., that

"for any integer $a \geq 0$, we have $f_a \mid f_{a(c+1)}$" holds.

Let $a \geq 0$ be any integer. Then, the induction hypothesis (i.e., our assumption that $P(c)$ holds) yields that $f_a \mid f_{ac}$. In other words, $f_{ac} = f_a p$ for some integer p. Now,

$$\begin{aligned} f_{a(c+1)} &= f_{ac+a} = f_{ac+(a-1)+1} \\ &= \underbrace{f_{ac}}_{=f_a p} f_{a-1} + f_{ac+1} f_a \qquad \left(\begin{array}{c} \text{by Theorem 1.8.3,} \\ \text{applied to } n = ac \text{ and } m = a-1 \end{array} \right) \\ &= f_a p f_{a-1} + f_{ac+1} f_a = f_a \cdot \underbrace{(p f_{a-1} + f_{ac+1})}_{\text{an integer}}. \end{aligned}$$

This immediately yields that $f_a \mid f_{a(c+1)}$. Thus, we have shown that for any integer $a \geq 0$, we have $f_a \mid f_{a(c+1)}$. In other words, we have proved that $P(c+1)$ holds. This completes the induction step, and thus the restated theorem is proved. Therefore, the original Theorem 1.8.5 is also proved.

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Is it? There is a subtle gap in our above argument. Can you find it?

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Can you? Don't look down just yet. The gap is somewhere above!

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This time, the theorem itself is correct, so you can't find the gap by tracing the proof through a case where the theorem is false. Though an example might be useful...

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No, we didn't misuse the principle of induction. The structure of the proof is fine. (Actually, we could have made our statements a bit shorter by fixing $a \geq 0$, but this wouldn't have made much of a difference.)

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The base case was fine, too.

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A computer, of course, would spot the problem.

If you tried to formalize the above proof in a computer language (e.g., Coq or Lean), you would run into a type mismatch error. Some statement has been proved for variables of a certain type, but is being used for variables of a different type. Very slightly different.

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The statement in question is Theorem 1.8.3. It is stated for one kind of variables, but we have used it for a slightly different kind.

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OK, I am spelling it out: Theorem 1.8.3 (i.e., the addition formula $f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$) has been stated and proved for all integers $n, m \geq 0$, but we have applied it to $n = ac$ and $m = a - 1$. For this to work, we need $ac \geq 0$ and $a - 1 \geq 0$. Now, $ac \geq 0$ is indeed satisfied (since $a \geq 0$ and $c \geq 0$), but $a - 1 \geq 0$ holds only if $a \geq 1$, which is not guaranteed. Thus, our use of Theorem 1.8.3 was illegal when $a = 0$. And indeed, if we apply Theorem 1.8.3 for $a = 0$, then we end up with an f_{-1} term, which is undefined. Even if you define f_{-1} appropriately (and there is a good definition; see homework set #1 Exercise 5), we have not proved Theorem 1.8.3 for negative n, m. So there is a gap in our proof. Can we fix it?

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Fortunately, we can: Our argument breaks down only in the case when $a = 0$, and we can just treat this case $a = 0$ manually, since it is an easy case. So we build a case distinction into our above induction step. Thus, the induction step takes the following form:

Induction step (corrected): Let $c \geq 0$ be an integer. We assume that $P(c)$ holds, i.e., that

"for any integer $a \geq 0$, we have $f_a \mid f_{ac}$" holds.

We must prove that $P(c+1)$ holds, i.e., that

"for any integer $a \geq 0$, we have $f_a \mid f_{a(c+1)}$" holds.

Let $a \geq 0$ be any integer. We must show that $f_a \mid f_{a(c+1)}$. We are in one of the following two cases:

Case 1: We have $a = 0$.

Case 2: We have $a \neq 0$.

In Case 1, we have $a = 0$, so that both f_a and $f_{a(c+1)}$ equal $f_0 = 0$, and thus $f_a \mid f_{a(c+1)}$ holds (since $0 \mid 0$). Thus, the divisibility $f_a \mid f_{a(c+1)}$ is proved in Case 1.

Now, consider Case 2. In this case, $a \neq 0$, so that $a \geq 1$ (because a is an integer and ≥ 0). Hence, $a - 1 \geq 0$. This will allow us to apply Theorem 1.8.3 to $n = ac$ and $m = a - 1$ in a few moments. The induction hypothesis (i.e., our assumption that $P(c)$ holds) yields that $f_a \mid f_{ac}$. In other words, $f_{ac} = f_a p$ for some integer p. Now,

$$\begin{aligned} f_{a(c+1)} &= f_{ac+a} = f_{ac+(a-1)+1} \\ &= \underbrace{f_{ac}}_{=f_a p} f_{a-1} + f_{ac+1} f_a \qquad \left(\begin{array}{c}\text{by Theorem 1.8.3,} \\ \text{applied to } n = ac \text{ and } m = a-1\end{array}\right) \\ &= f_a p f_{a-1} + f_{ac+1} f_a = f_a \cdot \underbrace{\left(p f_{a-1} + f_{ac+1}\right)}_{\text{an integer}}. \end{aligned}$$

This immediately yields that $f_a \mid f_{a(c+1)}$.

So we have proved $f_a \mid f_{a(c+1)}$ in both Cases 1 and 2. Therefore, $f_a \mid f_{a(c+1)}$ always holds.

Thus, $P(c+1)$ is proved. This completes the induction step, and thus the restated theorem is proved. Therefore, Theorem 1.8.5 is proved – correctly this time! □