

Math 221 Winter 2023 (Darij Grinberg): homework set 2

due date: Sunday 2023-01-29 at 11:59PM on gradescope (

<https://www.gradescope.com/courses/487830>).Please solve only **4 of the 6 exercises**.Recall that $\mathbb{N} = \{0, 1, 2, \dots\}$.**Exercise 1. (a)** Prove that

$$\prod_{i=2}^n \left(1 - \frac{1}{i^2}\right) = \frac{n+1}{2n}$$

for each positive integer n .**(b)** Find and prove a closed-form expression (i.e., no $\prod$ or $\sum$ signs) for

$$\prod_{i=2}^n \left(1 - \frac{1}{i}\right).$$

Exercise 2. Prove that

$$\prod_{i=1}^n (i! \cdot i^i) = n!^{n+1} \quad \text{for each } n \in \mathbb{N}.$$

The *floor* $\lfloor x \rfloor$ of a real number x means the largest integer that is smaller or equal to x . For instance, $\lfloor 6.2 \rfloor = 6$ and $\lfloor 7.7 \rfloor = 7$ and $\lfloor 8 \rfloor = 8$. (In other words, $\lfloor x \rfloor$ is what you get if you round x down. Beware: $\lfloor -1.3 \rfloor$ is -2 , not -1 .)

Exercise 3. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=1}^n \left\lfloor \frac{k}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n+1}{2} \right\rfloor.$$

Exercise 4. Let $(a_0, a_1, a_2, \dots)$ be a sequence of integers defined recursively by

$$\begin{aligned} a_0 &= 2, & a_1 &= 3, \\ a_n &= 3a_{n-1} - 2a_{n-2} & \text{for all } n \geq 2. \end{aligned}$$

Prove that $a_n = 2^n + 1$ for each $n \in \mathbb{N}$.

Now, we recall again the Fibonacci sequence $(f_0, f_1, f_2, \dots)$ that we got to know in Lecture 2. It is defined recursively by $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for each $n \geq 2$.

Exercise 5. (a) Let $k \in \mathbb{N}$. Show that

$$f_n^2 - f_{n+k}f_{n-k} = (-1)^{n-k} f_k^2 \quad \text{for every integer } n \geq k.$$

(b) Which exercise from homework set #1 does this generalize?

Exercise 6. Prove that every $n \in \mathbb{N}$ and $m \in \mathbb{N}$ satisfy

$$\sum_{k=0}^n \binom{n}{k} f_{m+k} = f_{m+2n}.$$