Math 221 Winter 2023 (Darij Grinberg): homework set 1
due date: Sunday 2023-01-22 at noon on gradescope (`https://www.gradescope.com/courses/487830`).
Please solve only **4 of the 6 exercises**.

We write $\mathbb{N}$ for the set of all nonnegative integers, i.e., for the set $\{0, 1, 2, \ldots\}$.

For each $n \in \mathbb{N}$, we define $n!$ (this is called the *factorial* of n, and is pronounced "n factorial") to be the **product** of the first n positive integers. That is, we set

$$n! = 1 \cdot 2 \cdot \cdots \cdot n.$$

Here (and everywhere else), we follow the convention that an empty product (i.e., a product with no factors) is 1 by definition. Thus, $0! = 1$ (being such an empty product). Here is a little table of factorials:

n	0	1	2	3	4	5	6	7	8
$n!$	1	1	2	6	24	120	720	5 040	40 320

.

It is clear that $n! = (n-1)! \cdot n$ for each positive integer n (because $n!$ is the product $1 \cdot 2 \cdot \cdots \cdot n$, whereas $(n-1)!$ is the product $1 \cdot 2 \cdot \cdots \cdot (n-1)$, which is the previous product without its last factor).

Exercise 1. **(a)** Prove that

$$1^3 + 2^3 + \cdots + n^3 = \left(\frac{n(n+1)}{2} \right)^2$$

for each $n \in \mathbb{N}$. (The left hand side here is the sum of the **cubes** of the first n positive integers.)

(b) Prove that

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n+1)! - 1$$

for each $n \in \mathbb{N}$.

(Meanwhile, there is no such simple formula for $1! + 2! + 3! + \cdots + n!$. Not every sum can be simplified!)

Exercise 2. Let q and d be two real numbers such that $q \neq 1$. Let $(a_0, a_1, a_2, \ldots)$ be a sequence of real numbers. Assume that

$$a_{n+1} = qa_n + d \qquad \text{for each } n \in \mathbb{N}. \tag{1}$$

Prove that

$$a_n = q^n a_0 + \frac{q^n - 1}{q - 1} d \qquad \text{for each } n \in \mathbb{N}. \tag{2}$$

Now, we recall again the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ that we got to know in Lecture 2. It is defined recursively by $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for each $n \geq 2$.

Exercise 3. **(a)** Show that

$$f_{n+1} f_{n-1} - f_n^2 = (-1)^n \qquad \text{for every positive integer } n.$$

(The word "Show" is a synonym for "Prove".)

(b) Show that

$$f_1^2 + f_2^2 + \cdots + f_n^2 = f_n f_{n+1} \qquad \text{for each } n \in \mathbb{N}.$$

(The left hand side here is the sum of the squares of the first n positive Fibonacci numbers.)

Let us now generalize the properties of the Fibonacci sequence that we started proving in Lecture 3:

Exercise 4. Let u and v be two real numbers. Let $(x_0, x_1, x_2, \ldots)$ be a sequence of real numbers such that $x_0 = 0$ and $x_1 = 1$ and

$$x_n = u x_{n-1} + v x_{n-2} \qquad \text{for each } n \geq 2.$$

(When $u = 1$ and $v = 1$, this is the Fibonacci sequence.) Prove that

$$x_{n+m+1} = v x_n x_m + x_{n+1} x_{m+1} \qquad \text{for all } n, m \in \mathbb{N}.$$

Now, let us extend the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ "to the left" by defining f_n not only for nonnegative integers n, but also for negative integers n. To do so, we simply rewrite the equation $f_n = f_{n-1} + f_{n-2}$ (which we used to recursively define the Fibonacci sequence) as $f_{n-2} = f_n - f_{n-1}$. This allows us to compute f_{n-2} from f_n and f_{n-1}. Thus, we can compute f_{-1} from f_1 and f_0, then compute f_{-2} from f_0 and f_{-1}, and so on:

$$\begin{aligned} f_{-1} &= f_1 - f_0 = 1 - 0 = 1; \\ f_{-2} &= f_0 - f_{-1} = 0 - 1 = -1; \\ f_{-3} &= f_{-1} - f_{-2} = 1 - (-1) = 2; \\ f_{-4} &= f_{-2} - f_{-3} = (-1) - 2 = -3; \\ &\ldots. \end{aligned}$$

Thus, we gradually extend the Fibonacci sequence to the left, obtaining a "two-sided sequence" $(\ldots, f_{-2}, f_{-1}, f_0, f_1, f_2, \ldots)$ that is "infinite in both directions". By virtue of its construction, it satisfies $f_n = f_{n-1} + f_{n-2}$ not only for all $n \geq 2$, but also for all integers n. However, a quick look at the first (say) 7 "extended" Fibonacci numbers to the left of f_0 reveals that they are not as new as they might seem: They are just copies of the positive Fibonacci numbers with signs. More precisely, it looks like we have

$$f_{-n} = (-1)^{n-1} f_n \qquad \text{for each } n \in \mathbb{N}. \tag{3}$$

Exercise 5. (a) Try to prove (3) directly by induction on n. (So the induction step involves assuming that $f_{-n} = (-1)^{n-1} f_n$ and proving that $f_{-(n+1)} = (-1)^n f_{n+1}$.) Does this work?

(b) Now, instead, try to prove the **stronger** claim that "$f_{-n} = (-1)^{n-1} f_n$ and $f_{-n+1} = (-1)^{n-2} f_{n-1}$ for each $n \in \mathbb{N}$" by induction on n. Does this work?

Finally, we return to the Tower of Hanoi puzzle, which we introduced in Lecture 1:

Exercise 6. Let $n \in \mathbb{N}$ and $k \in \{1, 2, \ldots, n\}$. In Lecture 1, we discussed a certain strategy for solving the Tower of Hanoi puzzle with n disks.

Prove that the k-th largest disk is moved exactly 2^{k-1} many times during this strategy.