Drexel University, Department of Mathematics

Math 332: Undergraduate Abstract Algebra II, Winter 2023: Midterm 3

Please solve **at most 3 of the 6 problems!** **No collaboration** is allowed on the midterm.

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1 Exercise 1

1.1 Problem

Let R be any ring. Let $n \geq 2$ be an integer. Let W be the subset

$$\{(a_1, a_2, \ldots, a_n) \in R^n \mid a_i - a_{i-1} = a_{i+1} - a_i \text{ for each } i \in \{2, 3, \ldots, n-1\}\}$$

of the left R-module R^n. (This set W consists of all vectors $(a_1, a_2, \ldots, a_n) \in R^n$ whose entries "form an arithmetic progression", i.e., satisfy $a_2 - a_1 = a_3 - a_2 = a_4 - a_3 = \cdots = a_n - a_{n-1}$.)

(a) Prove that W is an R-submodule of R^n.

(b) Let $a = (1, 1, \ldots, 1) \in R^n$ and $b = (1, 2, \ldots, n) \in R^n$. Prove that (a, b) is a basis of W.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let R be a commutative ring. Let $f \in R[x, y]$ be a polynomial in two variables x and y. Let $g = f[y, x]$. (This is the evaluation of f at y, x. In other words, g is the result of replacing each monomial $x^i y^j$ by $y^i x^j$ in f. For example, if $f = x^2 + 7xy - y$, then $g = y^2 + 7yx - x$.)

Prove that the difference $f - g$ is divisible by $x - y$ in the ring $R[x, y]$.

2.2 Hint

Use linearity to reduce the general case to the case when f is a single monomial.

2.3 Solution

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3 Exercise 3

3.1 Problem

Let R be any commutative ring. Let $n \in \mathbb{N}$.

(a) Prove that the remainder obtained when dividing x^n by $(x-1)^2$ (in the polynomial ring $R[x]$) is $nx - n + 1$.

(b) In terms of the Fibonacci numbers, find the quotient and the remainder obtained when dividing x^n by $x^2 - x - 1$.

3.2 Hint

Compute them (e.g.) for $n = 10$, and prove the pattern you discover.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let F be a finite field.

(a) Prove that

$$x^{|F|} - x = \prod_{u \in F} (x - u) \qquad \text{in the polynomial ring } F[x].$$

(b) Prove that the product of all nonzero elements of F equals -1_F.

4.2 Remark

Part **(a)** generalizes the formula $x^p - x = \prod_{u \in \mathbb{Z}/p} (x - u) \in (\mathbb{Z}/p)[x]$ for prime numbers p. Part **(b)** generalizes Wilson's theorem.

4.3 Solution

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Let $p \in \mathbb{Q}[x]$ be a polynomial of degree $\leq n$ such that

$$p[i] = 2^i \qquad \text{for all } i \in \{0, 1, \ldots, n\}.$$

Find $p[n+1]$.

5.2 Hint

Remember Lagrange interpolation, but don't forget the binomial formula either.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let p be a prime. Let $k \in \mathbb{N}$. Prove that the integer $0^k + 1^k + \cdots + (p-1)^k = \sum_{j=0}^{p-1} j^k$ is divisible by p if and only if k is not a positive multiple of $p - 1$.

6.2 Hint

This relies on Section 4.3.5 of the text.

6.3 Solution

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