Drexel University, Department of Mathematics

Math 332: Undergraduate Abstract Algebra II, Winter 2023: Homework 4

Please solve **at most 3 of the 6 problems!**

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1 Exercise 1

1.1 Problem

In this exercise, we shall see how idempotent central elements are responsible for rings decomposing as direct products.

Let R be a ring, and let e be an idempotent central element of R. (Recall: "Central" means that $eb = be$ for all $b \in R$.)

Exercise 4 **(a)** on homework set #1 shows that $1 - e \in R$ is again idempotent.

- **(a)** Show that $1 - e$ is furthermore central.
- **(b)** Show that the principal ideal eR is itself a ring, with addition and multiplication inherited from R and with zero 0_R and with unity e. (This almost makes eR a subring of R, but not quite, since a subring would have to have unity 1_R.)
- **(c)** Show that the same holds for the principal ideal $(1 - e) R$ (except that its unity will be $1 - e$ instead of e).

(d) Consider the map

$$f : (eR) \times ((1-e)R) \to R,$$
$$(a, b) \mapsto a + b.$$

Prove that this map f is a ring isomorphism.

1.2 Remark

Part **(d)** of this exercise shows that if a ring R has an idempotent central element e, then R can be decomposed (up to isomorphism) as a direct product $A \times B$ of two rings A and B (namely, $A = eR$ and $B = (1-e)R$). If e is not one of the two trivial idempotents 0 and 1, then these two rings A and B will be nontrivial, so the decomposition really deserves its name.[1]

Conversely, any direct product of two nontrivial rings has nontrivial central idempotents: If R and S are two rings, then $(1_R, 0_S)$ and $(0_R, 1_S)$ are two idempotent central elements of the direct product $R \times S$.

1.3 Solution

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2 Exercise 2

2.1 Problem

Let R be any nontrivial ring. Consider the ideals

$$I := \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in R \right\},$$
$$J := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in R \right\},$$
$$K := \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \right\}$$

of the upper-triangular matrix ring $R^{2 \leq 2}$ defined in Exercise 2 of homework set #1. Note that $K = I \cap J$.

(a) Prove that the ideals I and J are comaximal (i.e., we have $I + J = R^{2 \leq 2}$).

(b) Prove that $IJ = \{0\}$.

(c) Prove that $JI = K$.

[1] As an example, take $R = \mathbb{Z}/6\mathbb{Z}$, and let e be the idempotent element $\overline{3} = 3 + 6\mathbb{Z}$ of R (this is idempotent since $3^2 = 9 \equiv 3 \mod 6$ and thus $\overline{3}^2 = \overline{3^2} = \overline{3}$). Then, $eR = \left\{\overline{0}, \overline{3}\right\} \cong \mathbb{Z}/2\mathbb{Z}$ and $(1-e)R = \left\{\overline{0}, \overline{2}, \overline{4}\right\} \cong \mathbb{Z}/3\mathbb{Z}$. Hence, the ring isomorphism $R \cong (eR) \times ((1-e)R)$ becomes the ring isomorphism $\mathbb{Z}/6\mathbb{Z} \cong (\mathbb{Z}/2\mathbb{Z}) \times (\mathbb{Z}/3\mathbb{Z})$ that we have seen in Lecture 13 (as an instance of the Chinese Remainder Theorem).

2.2 Remark

This exercise illustrates that $I \cap J = IJ$ does not always hold in the noncommutative case.

2.3 Solution

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3 Exercise 3

3.1 Problem

Consider the quotient ring $\mathbb{Q}^{3\leq 3}/\mathbb{Q}^{3<<3}$ that was constructed in Lecture 9. Recall that we write its elements (which are residue classes of upper-triangular 3×3-matrices modulo the ideal $\mathbb{Q}^{3<<3}$) as "partly determined" upper-triangular 3×3-matrices with a question mark in the top-right corner.

(a) Prove that the map

$$f : \mathbb{Q}^{3\leq 3} \to \mathbb{Q}^{2\times 2},$$
$$\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & g \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$$

is a ring morphism.

(b) Prove that this morphism f satisfies $f\left(\mathbb{Q}^{3<<3}\right) = 0$.

(c) Use the universal property of quotient rings to conclude that there is a ring morphism

$$f' : \mathbb{Q}^{3\leq 3}/\mathbb{Q}^{3<<3} \to \mathbb{Q}^{2\times 2},$$
$$\begin{pmatrix} a & b & ? \\ 0 & d & e \\ 0 & 0 & g \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

(d) Use a similar reasoning to prove the existence of a ring morphism

$$F' : \mathbb{Q}^{3\leq 3}/\mathbb{Q}^{3<<3} \to \mathbb{Q}^{4\times 4},$$
$$\begin{pmatrix} a & b & ? \\ 0 & d & e \\ 0 & 0 & g \end{pmatrix} \mapsto \begin{pmatrix} a & b & 0 & 0 \\ 0 & d & 0 & 0 \\ 0 & 0 & d & e \\ 0 & 0 & 0 & g \end{pmatrix},$$

which is furthermore injective.

(e) Conclude that the ring $\mathbb{Q}^{3\leq 3}/\mathbb{Q}^{3<<3}$ is isomorphic to a subring of $\mathbb{Q}^{4\times 4}$.

[When proving that a map is a ring morphism, feel free to only check the "respects multiplication" axiom, as the other axioms are essentially obvious in the present context. Some parts of this exercise can be solved in 1 sentence.]

3.2 Remark

Part **(e)** shows that if you want to represent the elements of $\mathbb{Q}^{3\leq 3}/\mathbb{Q}^{3<<3}$ as actual honest matrices (i.e., not matrices with question marks or other kinds of residue classes), then you can do it using 4×4-matrices (by duplicating the second entry on the diagonal). Doing it by 3×3-matrices alone does not work (see `https://mathoverflow.net/questions/439996` for a proof).

More generally, "partly determined" upper-triangular $n \times n$-matrices with a question mark in their top-right corner can be represented as honest $(2n-1) \times (2n-1)$-matrices (but not as matrices of any smaller size).

3.3 Solution

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4 Exercise 4

4.1 Problem

Let R be a ring. Prove the following:

(a) If I and J are two ideals of R, then IJ is an ideal of R as well.

(b) The set of all ideals of R is a monoid with respect to the binary operation $\cdot$, with neutral element $R = 1R$. That is, we have

$$(IJ)\,K = I\,(JK) \qquad \text{for any three ideals } I, J, K \text{ of } R;$$
$$IR = RI = I \qquad \text{for any ideal } I \text{ of } R.$$

4.2 Remark

This is part of Proposition 1.11.2 in Lecture 12 (which you cannot use without proof, of course).

I recommend convincing yourself that the rest of said proposition is true as well (but don't bother writing it down unless you want to). Most of the work required here is book-keeping. Note that it is easier to talk abstractly about sums of (I, J)-products than to write them out as $i_1j_1 + i_2j_2 + \cdots + i_kj_k$. For the proof of $(IJ)\,K = I\,(JK)$, a good approach is to first show that any (IJ, K)-product belongs to $I\,(JK)$.

4.3 Solution

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5 Exercise 5

5.1 Problem

Let R be a ring. Let I and J be two comaximal ideals of R. Let $n \in \mathbb{N}$ and $m \in \mathbb{N}$. Prove that I^n and J^m, too, are comaximal.

5.2 Solution

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6 Exercise 6

6.1 Problem

Let R be a ring. Let $n \in \mathbb{N}$.

For each subset I of R, let $I^{n\times n}$ be the subset

$$\left\{A \in R^{n\times n} \mid \text{all entries of } A \text{ belong to } I\right\}$$

of the matrix ring $R^{n\times n}$.

Prove the following:

(a) If I is an ideal of R, then $I^{n\times n}$ is an ideal of the matrix ring $R^{n\times n}$.

(b) Any ideal of $R^{n\times n}$ has the form $I^{n\times n}$ for some ideal I of R.

6.2 Remark

In particular, if F is a field, then the matrix ring $F^{n\times n}$ has only two ideals, namely $\{0\}$ and the whole $F^{n\times n}$ (where 0 stands for the zero matrix). This is because the field F has only two ideals ($\{0\}$ and F).

(In contrast, the matrix ring $F^{n\times n}$ has many more left ideals and right ideals – i.e., almost-ideals that satisfy only one half of the second ideal axiom.[2])

6.3 Hint

For each $i, j \in \{1, 2, \ldots, n\}$, let $E_{i,j} \in R^{n\times n}$ be the (i, j)*-th elementary matrix* – i.e., the $n \times n$-matrix whose (i, j)-th entry is 1 and whose all remaining entries are 0. What happens when you multiply a given matrix $A \in R^{n\times n}$ by $E_{i,j}$ from the left or from the right? I.e., how can you describe the matrices $E_{i,j}A$ and $AE_{i,j}$? This exercise will make you truly appreciate elementary matrices for all the operations that they can perform on a matrix.

[2]For example, a *left ideal* L of a ring R must satisfy $a\ell \in L$ for all $a \in R$ and $\ell \in L$, but not necessarily $\ell a \in L$.

6.4 Solution

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