Drexel University, Department of Mathematics

Math 332: Undergraduate Abstract Algebra II, Winter 2023: Homework 3

Please solve **at most 3 of the 6 problems!**

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1 Exercise 1

1.1 Problem

Let R and S be two rings. Let $f : R \to S$ be a ring morphism.

Which of the following two claims are true? (Prove the true ones and give counterexamples to the false ones.)

(a) If I is an ideal of R, then $f(I) := \{f(i) \mid i \in I\}$ is an ideal of S.

(b) If K is an ideal of S, then $f^{-1}(K) := \{i \in I \mid f(i) \in K\}$ is an ideal of R.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let R be any ring. Recall that $R^{n \leq n}$ denotes the ring of all upper-triangular $n \times n$-matrices with entries in R. In particular,

$$R^{2 \leq 2} = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in R \right\}.$$

Define four subsets I, J, K, L of $R^{2 \leq 2}$ by

$$I := \left\{ \begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} \mid b, d \in R \right\};$$
$$J := \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in R \right\};$$
$$K := \left\{ \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix} \mid b \in R \right\};$$
$$L := \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \mid a, d \in R \right\}.$$

Prove that I, J and K are ideals of $R^{2 \leq 2}$, but L is not (unless R is trivial).

Feel free to omit the proofs of the "contains zero" and "is closed under addition" ideal axioms. (These axioms are satisfied for all four of I, J, K, L for pretty obvious reasons.)

2.2 Solution

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3 Exercise 3

3.1 Problem

Let R be a ring. An element $a \in R$ is said to be *nilpotent* if there exists an $n \in \mathbb{N}$ such that $a^n = 0$. (For example, the residue class $\overline{6}$ in $\mathbb{Z}/8$ is nilpotent, since its 3-rd power is $\overline{0}$.)

(a) If a and b are two nilpotent elements of R satisfying $ab = ba$, then prove that $a + b$ is nilpotent as well.

(b) Find a counterexample to part **(a)** if we don't assume $ab = ba$.

(c) Assume that the ring R is commutative. Let N be the set of all nilpotent elements of R. Prove that N is an ideal of R.

3.2 Remark

The ideal N defined in part **(c)** is called the *nilradical* of R.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let n be a positive integer with prime factorization $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, where $p_1, p_2, \ldots, p_k$ are distinct primes and $a_1, a_2, \ldots, a_k$ are positive integers. Let R be the ring $\mathbb{Z}/n$. Prove that the nilradical[1] of R is the principal ideal $\overline{p_1 p_2 \cdots p_k} R$.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let R be the ring of all real numbers of the form $a + b\sqrt{5}$ with $a, b \in \mathbb{Z}$. (You can take for granted that this is indeed a ring; the proof is analogous to the proof for $\mathbb{S}$ in Lecture 2.)

(a) Is the quotient ring $R/(2R)$ a field?

(b) Is the quotient ring $R/(3R)$ a field?

(c) Prove that the quotient ring $R/(5R)$ is not a field, and in fact the residue class $\overline{\sqrt{5}}$ in this quotient is nilpotent. (See Exercise 3 above for the definition of "nilpotent".)

5.2 Solution

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6 Exercise 6

6.1 Problem

Let R be the ring $\mathbb{Z}[i]$ of Gaussian integers. Let $z = 1 + 2i \in R$.

[1] See the remark after Exercise 3 for the definition of this notion.

(a) Prove that the quotient ring $R/(zR)$ has exactly 5 elements, namely $\overline{0}$, $\overline{1}$, $\overline{2}$, $\overline{3}$, $\overline{4}$. (This requires, in particular, showing that these 5 elements are distinct.)

(b) Which of these five elements is $\overline{12+7i}$?

6.2 Hint

Proceed as in Lecture 8. In particular, first show that $5 \in zR$.

6.3 Solution

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