Drexel University, Department of Mathematics

Math 332: Undergraduate Abstract Algebra II, Winter 2023: Homework 2

Please solve **at most 3 of the 6 problems!**[*]

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1 Exercise 1

1.1 Problem

Let R be a commutative ring in which $2 \cdot 1_R = 0_R$. (Examples of such rings are $\mathbb{Z}/2$ or polynomial rings over $\mathbb{Z}/2$.)

Prove that the set of all idempotent elements $a \in R$ is a subring of R.

1.2 Solution

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[*] I recommend solving as many problems as you can and wish, but I will only grade and score 3 solutions per submission (and if you submit more, I get to pick which ones I grade).
Results stated in class, and claims of previous problems (even if you did not solve these previous problems), can be used without proof. For example, in solving Problem 5, you can use the result of Problem 2 without proof.
I expect approximately the level of detail that I give in class. Purely straightforward arguments (like checking the ring axioms for a direct product of rings) need not be spelled out; I only expect a note of their necessity.

2 Exercise 2

2.1 Problem

Let R be a ring. Let $a, b \in R$ be such that $a + b$ is central[1]. Prove that $ab = ba$.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let R be a ring. Let $a, b \in R$ be such that ab is central. Prove that $(ab)^n = a^n b^n$ for all $n \in \mathbb{N}$.

3.2 Solution

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4 Exercise 4

4.1 Problem

Let R be the matrix ring $\mathbb{R}^{2\times 2}$. In this ring R, consider the two matrices

$$A := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad \text{and} \qquad B := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

(a) Describe the centralizer[2] $Z_R(\{A, B\})$.

(b) Describe the centralizer $Z_R(\{A + B\})$.

(c) Describe the centralizer $Z_R(\{A - B\})$.

You **do not need to give proofs** in this exercise if your claims are correct.

[1] An element $c \in R$ is said to be *central* if it satisfies $cx = xc$ for all $x \in R$. See Subsection 2.3.5 of the notes for more.

[2] See Subsection 2.3.5 in the notes for a definition.

4.2 Solution

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5 Exercise 5

5.1 Problem

In Exercise 2.3.6 in the notes, a commutative subring $\mathcal{F}$ of the matrix ring $\mathbb{Z}^{2\times 2}$ was constructed. This ring $\mathcal{F}$ consists of all matrices of the form

$$\begin{pmatrix} b & a \\ a & a+b \end{pmatrix} \qquad \text{with } a, b \in \mathbb{Z}.$$

(You can take all claims of Exercise 2.3.6 for granted; you can also find its solution on my website[3].)

(a) Prove that this ring $\mathcal{F}$ is an integral domain.

(b) Let $\mathcal{F}_{\mathbb{Q}}$ be the ring defined just like $\mathcal{F}$, but using $\mathbb{Q}$ instead of $\mathbb{Z}$ (that is, by replacing "$a, b \in \mathbb{Z}$" with "$a, b \in \mathbb{Q}$" in the definition of $\mathcal{F}$). Is $\mathcal{F}_{\mathbb{Q}}$ an integral domain?

(c) Let $\mathcal{F}_{\mathbb{R}}$ be the ring defined just like $\mathcal{F}$, but using $\mathbb{R}$ instead of $\mathbb{Z}$ (that is, by replacing "$a, b \in \mathbb{Z}$" with "$a, b \in \mathbb{R}$" in the definition of $\mathcal{F}$). Is $\mathcal{F}_{\mathbb{R}}$ an integral domain?

5.2 Hint

For part **(a)**, argue that the determinant

$$\det \begin{pmatrix} b & a \\ a & a+b \end{pmatrix} = -a^2 + ab + b^2 = \frac{5b^2 - (2a-b)^2}{4}$$

of any nonzero matrix in $\mathcal{F}$ is nonzero, since $\sqrt{5}$ is irrational. Now recall that $\det(AB) = \det A \cdot \det B$ for any $A, B \in \mathbb{R}^{2\times 2}$.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let p be a prime such that $p > 3$. Prove that $2^{p-2} + 3^{p-2} + 6^{p-2} \equiv 1 \mod p$.

[3] It's the solution to Exercise 5 in my Spring 2019 Math 4281 class, available at `https://www.cip.ifi.lmu.de/~grinberg/t/19s/mt2s.pdf` .

6.2 Hint

First, show that $u^{p-2} = \frac{1}{u}$ for every nonzero $u \in \mathbb{Z}/p$. Then, recall the formula $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

6.3 Solution

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