Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2023: Homework 7
due 2023-05-26 at 11:59 PM
Please solve 3 of the 7 problems!

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1 Exercise 1

1.1 Problem

Let $D = (V, A, \psi)$ be a multidigraph. Assume that $V = \{1, 2, \ldots, n\}$ for some positive integer n. Let L be the Laplacian of D.

Prove the following:

(a) If r and s are two vertices of D, then

$$(\text{\# of spanning arborescences of } D \text{ rooted to } r) = (-1)^{r+s} \det\left(L_{\sim r, \sim s}\right).$$

(b) For each $r \in V$, we let $\tau(D, r)$ be the # of spanning arborescences of D rooted to r. Let f be the row vector $(\tau(D,1), \tau(D,2), \ldots, \tau(D,n))$. Then, $fL = 0$ (the zero vector).

1.2 Hint

Part **(a)** generalizes the Matrix-Tree Theorem (which is its particular case for $s = r$). However, both parts can actually be derived using the properties of L we know from class (without any new combinatorial input).

1.3 Solution

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2 Exercise 2

2.1 Problem

Let n be a positive integer. Let $N = \{1, 2, \ldots, n\}$. A map $f : N \to N$ is said to be *n-potent* if each $i \in N$ satisfies $f^{n-1}(i) = n$. (As usual, f^k denotes the k-fold composition $f \circ f \circ \cdots \circ f$.)

Prove that the # of n-potent maps $f : N \to N$ is n^{n-2}.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let $n = 2m + 1 > 2$ be an odd integer. Let e be an edge of the (undirected) complete graph K_n. Prove that the # of Eulerian circuits of K_n that start with e is a multiple of $(m-1)!^n$.

3.2 Hint

Argue that each Eulerian circuit of K_n is a Eulerian circuit of a unique balanced tournament. Here, a "balanced tournament" means a balanced digraph obtained from K_n by orienting each edge.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let $G = (V, E, \varphi)$ be a multigraph. Let L be the Laplacian of the digraph G^{bidir}. Prove that L is positive semidefinite.

4.2 Hint

Write L as $N^T N$, where N or N^T is some matrix you have seen before.

Note that the statement is not true if we replace G^{bidir} by an arbitrary digraph D.

4.3 Solution

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5 Exercise 5

5.1 Problem

Let n be a positive integer. Let Q_n be the n-hypercube graph (as defined in Lecture 6). Recall that its vertex set is the set $V := \{0,1\}^n$ of length-n bitstrings, and that two vertices are adjacent if and only if they differ in exactly one bit. Our goal is to compute the # of spanning trees of Q_n.

Let D be the digraph Q_n^{bidir}. Let L be the Laplacian of D. We regard L as a $V \times V$-matrix (i.e., as a $2^n \times 2^n$-matrix whose rows and columns are indexed by bitstrings in V).

We shall use the notation a_i for the i-th entry of a bitstring a. Thus, each bitstring $a \in V$ has the form $a = (a_1, a_2, \ldots, a_n)$. (We shall avoid the shorthand notation $a_1 a_2 \cdots a_n$ here, as it could be mistaken for an actual product.)

For any two bitstrings $a, b \in V$, we define the number $\langle a, b \rangle$ to be the integer $a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$.

(a) Prove that every bitstring $a \in V$ satisfies

$$\sum_{b \in V} (-1)^{\langle a,b \rangle} = \begin{cases} 2^n, & \text{if } a = \mathbf{0}; \\ 0, & \text{otherwise.} \end{cases}$$

Here, $\mathbf{0}$ denotes the bitstring $(0, 0, \ldots, 0) \in V$.

Now, define a further $V \times V$-matrix G by requiring that its (a, b)-th entry is

$$G_{a,b} = (-1)^{\langle a,b \rangle} \qquad \text{for any } a, b \in V.$$

Furthermore, define a diagonal $V \times V$-matrix D by requiring that its (a, a)-th entry is

$$\begin{aligned} D_{a,a} &= 2 \cdot (\# \text{ of } i \in \{1, 2, \ldots, n\} \text{ such that } a_i = 1) \\ &= 2 \cdot (\text{the number of 1s in } a) \qquad \text{for any } a \in V \end{aligned}$$

(and its off-diagonal entries are 0).

Prove the following:

(b) We have $G^2 = 2^n \cdot I$, where I is the identity $V \times V$-matrix.

(c) We have $GLG^{-1} = D$.

(d) The eigenvalues of L are $2k$ for all $k \in \{0, 1, \ldots, n\}$, and each eigenvalue $2k$ appears with multiplicity $\dbinom{n}{k}$.

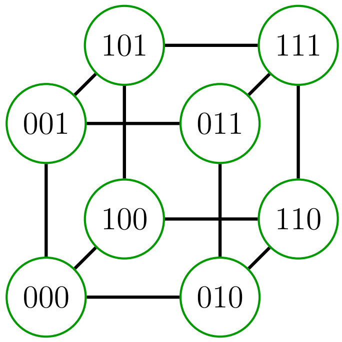
(e) The $\#$ of spanning trees of Q_n is

$$\frac{1}{2^n} \prod_{k=1}^{n} (2k)^{\binom{n}{k}}.$$

5.2 Remark

As an example, here is the case $n = 3$. In this case, the graph Q_n looks as follows:

$Q_3 =$



The matrices L, G and D are

$$L = \begin{pmatrix} 3 & -1 & -1 & 0 & -1 & 0 & 0 & 0 \\ -1 & 3 & 0 & -1 & 0 & -1 & 0 & 0 \\ -1 & 0 & 3 & -1 & 0 & 0 & -1 & 0 \\ 0 & -1 & -1 & 3 & 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 3 & -1 & -1 & 0 \\ 0 & -1 & 0 & 0 & -1 & 3 & 0 & -1 \\ 0 & 0 & -1 & 0 & -1 & 0 & 3 & -1 \\ 0 & 0 & 0 & -1 & 0 & -1 & -1 & 3 \end{pmatrix},$$

$$G = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \end{pmatrix},$$

where the rows and the columns are ordered by listing the eight bitstrings $a \in V$ in the order $000, 001, 010, 011, 100, 101, 110, 111$.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let G be a loopless multigraph. Recall that a *trail* means a walk whose edges are distinct (but whose vertices are not necessarily distinct). Let u and v be two vertices of G. As usual, "trail from u to v" means "trail that starts at u and ends at v".

Prove that

$$\begin{aligned}&(\text{the number of trails from } u \text{ to } v \text{ in } G)\\ \equiv\,&(\text{the number of paths from } u \text{ to } v \text{ in } G) \mod 2.\end{aligned}$$

6.2 Hint

Try to pair up the non-path trails into pairs. Make sure to prove that this pairing is well-defined (i.e., each non-path trail $\mathbf{t}$ has exactly one partner, which is not itself, and that $\mathbf{t}$ is the designated partner of its partner!).

6.3 Solution

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7 Exercise 7

7.1 Problem

Let T be a tree. Let w be any vertex of T. Prove that T has at least $\deg w$ many leaves.

7.2 Solution

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