Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2023: Homework 5 due 2023-05-11 at 11:59 PM

Please solve 3 of the 6 problems!

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1 Exercise 1

1.1 Problem

Let $D = (V, A, \psi)$ be a multidigraph.

For two vertices u and v of D, we shall write $u \overset{*}{\to} v$ if there exists a path from u to v.

A *root* of D means a vertex $u \in V$ such that each vertex $v \in V$ satisfies $u \overset{*}{\to} v$.

A *common ancestor* of two vertices u and v means a vertex $w \in V$ such that $w \overset{*}{\to} u$ and $w \overset{*}{\to} v$.

Assume that D has at least one vertex. Prove that D has a root if and only if every two vertices in D have a common ancestor.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let G be a multigraph that has no loops. Assume that there exists a vertex u of G such that

for each vertex v of G, there is a **unique** path from u to v in G.

Prove that G is a tree.

2.2 Remark

Notice the quantifiers used here: $\exists u \forall v$. This differs from the $\forall u \forall v$ in Statement T2 of the tree equivalence theorem (Theorem 1.2.4 in Spring 2022 Lecture 13).

2.3 Solution

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3 Exercise 3

3.1 Problem

Let F be any field[1].

Let $G = (V, E, \varphi)$ be a multigraph, where $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

For each edge $e \in E$, we construct a column vector $\chi_e \in F^n$ (that is, a column vector with n entries) as follows:

- If e is a loop, then we let χ_e be the zero vector.
- Otherwise, we let u and v be the two endpoints of e, and we let χ_e be the column vector that has a 1 in its u-th position, a -1 in its v-th position, and 0s in all other positions. (This depends on which endpoint we call u and which endpoint we call v, but we just make some choice and stick with it. The result will be true no matter how we choose.)

Let M be the $n \times |E|$-matrix over F whose columns are the column vectors χ_e for all $e \in E$ (we order them in some way; the exact order doesn't matter). Prove that

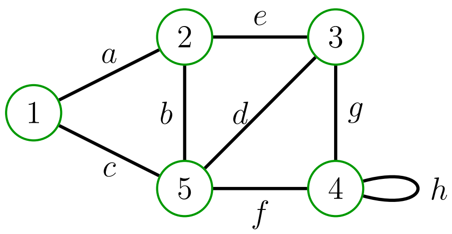
$$\operatorname{rank} M = |V| - \operatorname{conn} G.$$

(Recall that $\operatorname{conn} G$ denotes the number of components of G.)

[1] If you find it more convenient, you can assume that $F = \mathbb{R}$ or $F = \mathbb{C}$.

3.2 Remark

Here is an example: Let G be the multigraph



(so that $n = 5$). Then, if we choose the endpoints of b to be 2 and 5 in this order, then we have $\chi_b = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$. (Choosing them to be 5 and 2 instead, we would obtain $\chi_b = \begin{pmatrix} 0 \\ -1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$.) If we do the same for all edges of G (that is, we choose the smaller endpoint as u and the larger endpoint as v), and if we order the columns so that they correspond to the edges a, b, c, d, e, f, g, h from left to right, then the matrix M comes out as follows:

$$M = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & -1 & -1 & -1 & 0 & -1 & 0 & 0 \end{pmatrix}.$$

It is easy to see that $\operatorname{rank} M = 4$, which is precisely $|V| - \operatorname{conn} G$.

Another way of putting the claim of the exercise is that the span of the vectors χ_e for all $e \in E$ has dimension $|V| - \operatorname{conn} G$.

Topologists will recognize the matrix M as (a matrix that represents) the boundary operator $\partial : C_1(G) \to C_0(G)$, where G is viewed as a CW-complex.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let $G = (V, E, \varphi)$ be a connected multigraph such that $|E| \geq |V|$. Show that there exists an injective map $f : V \to E$ such that for each vertex $v \in V$, the edge $f(v)$ contains v.

(In other words, show that we can assign to each vertex an edge that contains this vertex in such a way that no edge is assigned twice.)

4.2 Solution

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5 Exercise 5

5.1 Problem

Let G be a connected multigraph. Let T_1 and T_2 be two spanning trees of G.

Prove the following:

(a) For any $e \in \operatorname{E}(T_1) \setminus \operatorname{E}(T_2)$, there exists an $f \in \operatorname{E}(T_2) \setminus \operatorname{E}(T_1)$ with the property that replacing e by f in T_1 (that is, removing the edge e from T_1 and adding the edge f) results in a spanning tree of G.

(b) For any $f \in \operatorname{E}(T_2) \setminus \operatorname{E}(T_1)$, there exists an $e \in \operatorname{E}(T_1) \setminus \operatorname{E}(T_2)$, with the property that replacing e by f in T_1 (that is, removing the edge e from T_1 and adding the edge f) results in a spanning tree of G.

5.2 Remark

The two parts look very similar, but (to my knowledge) their proofs are not.

5.3 Solution

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6 Exercise 6

6.1 Problem

If G is any graph (i.e., simple graph or multigraph), then P_G shall denote the polynomial

$$\sum_{D \text{ is a dominating set of } G} x^{|D|}$$

in the indeterminate x (with integer coefficients). Here, a dominating set of a multigraph G is defined to be a dominating set of its underlying simple graph G^{simp}. We call P_G the *domination polynomial* of G. (Its x^k-coefficient is the number of dominating sets of G that have size k.)

Prove the following:

(a) The domination polynomial P_{K_n} of the complete graph K_n is $(1+x)^n - 1$ for each positive integer n.

(b) Ignore this part – it is false!

Let G be a graph. Let ℓ be a vertex of G that has exactly one neighbor. Let p be this neighbor. Assume that $\ell \neq p$.

Let A be the set $\{\ell, p\}$, and let B be the set consisting of p and all neighbors of p. (Note that $A \subseteq B \subseteq \operatorname{V}(G)$.)

If U is any set of vertices of G, then $G \setminus U$ shall mean the graph obtained from G by removing all the vertices in U (and all the edges that contain any of these vertices).

Then,

$$P_G = (1+x)\, x P_{G \setminus B} + x P_{G \setminus A}.$$

(c) If G is a forest, then

$$\sum_{D \text{ is a dominating set of } G} (-1)^{|D|} = \pm 1.$$

6.2 Remark

I would be particularly delighted to see a direct proof of part **(c)** (not using **(b)**), even more so if it explicitly describes the sign of the ± 1. (To my knowledge, such a description is not currently known, nor is a direct proof.)

6.3 Solution

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