Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2023: Homework 4
due 2023-05-04 at 11:59 PM
Please solve 3 of the 6 problems!

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1 Exercise 1

1.1 Problem

(a) Let $G = (V, E)$ be a simple graph, and let u and v be two distinct vertices of G that are not adjacent. Let $n = |V|$. Assume that $\deg u + \deg v \geq n$. Let $G' = (V, E \cup \{uv\})$ be the simple graph obtained from G by adding a new edge uv. Assume that G' has a hamc. Prove that G has a hamc.

(b) Does this remain true if we replace "hamc" by "hamp"?

1.2 Solution

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2 Exercise 2

2.1 Problem

Let D be a multidigraph with at least one vertex. Prove the following:

(a) If each vertex v of D satisfies $\deg^+ v > 0$, then D has a cycle.

(b) If each vertex v of D satisfies $\deg^+ v = \deg^- v = 1$, then each vertex of D belongs to exactly one cycle of D. Here, two cycles are considered to be identical if one can be obtained from the other by cyclic rotation.

2.2 Solution

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3 Exercise 3

3.1 Problem

Prove the directed Euler–Hierholzer theorem:

Let D be a weakly connected multidigraph. Then:

(a) The multidigraph D has an Eulerian circuit if and only if each vertex v of D satisfies $\deg^+ v = \deg^- v$.

(b) The multidigraph D has an Eulerian walk if and only if all but two vertices v of D satisfy $\deg^+ v = \deg^- v$, and the remaining two vertices v satisfy $\left|\deg^+ v - \deg^- v\right| \leq 1$.

3.2 Solution

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4 Exercise 4

4.1 Problem

(a) Let $D = (V, A, \psi)$ be a multidigraph, where $V = \{1, 2, \ldots, n\}$ for some $n \in \mathbb{N}$.

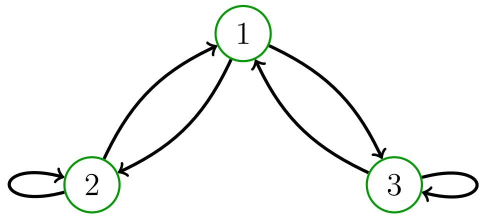
If M is any matrix, and if i and j are two positive integers, then $M_{i,j}$ shall denote the (i, j)-th entry of M (that is, the entry of M in the i-th row and the j-th column).

Let C be the $n \times n$-matrix (with real entries) defined by

$$C_{i,j} = (\text{the number of all arcs } a \in A \text{ with source } i \text{ and target } j) \qquad \text{for all } i, j \in V.$$

Let $k \in \mathbb{N}$, and let $i, j \in V$. Prove that $\left(C^k\right)_{i,j}$ equals the number of all walks of D having starting point i, ending point j and length k.

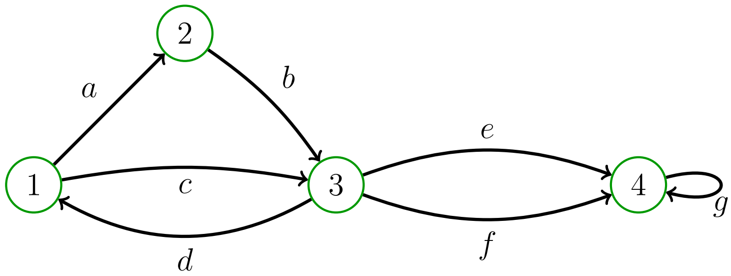
(b) Let E be the following multidigraph:

$E =$ 

Let $n \in \mathbb{N}$. Compute the number of walks from 1 to 1 having length n.

4.2 Remark

The matrix C in part **(a)** of this problem is known as the *adjacency matrix* of D. For example, if the multidigraph is

$D =$ 

then its adjacency matrix is

$$\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The adjacency matrix of a multidigraph D determines D up to the identities of the arcs, and thus is often used as a convenient way to encode a multidigraph.

4.3 Solution

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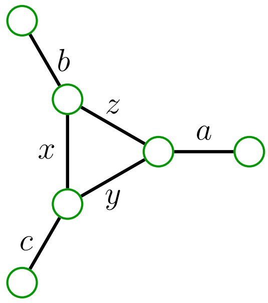
5 Exercise 5

5.1 Problem

Let $G = (V, E, \varphi)$ be a connected multigraph with $2m$ edges, where $m \in \mathbb{N}$. A set $\{e, f\}$ of two distinct edges will be called a *friendly couple* if e and f have at least one endpoint in common. Prove that the edge set of G can be decomposed into m disjoint[1] friendly couples (i.e., there exist m disjoint friendly couples $\{e_1, f_1\}, \{e_2, f_2\}, \ldots, \{e_m, f_m\}$ such that $E = \{e_1, f_1, e_2, f_2, \ldots, e_m, f_m\}$).

[1]"Disjoint" means "disjoint as sets" – i.e., having no edges in common.

[**Example:** Here is a graph with an even number of edges:



One possible decomposition into disjoint friendly pairs is $\{a, y\}, \{b, z\}, \{c, x\}$.]

5.2 Hint

Induct on $|E|$. Pick a vertex v of degree > 1 and consider the components of $G \setminus v$.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let D be a simple digraph with n vertices and a arcs. Assume that D has no loops, and that we have $a > n^2/2$. Prove that D has a cycle of length 3.

6.2 Remark

Note that this is both an analogue and a generalization (why?) of Mantel's theorem.

6.3 Solution

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