Drexel University, Department of Mathematics

Math 530: Graph Theory, Spring 2023: Homework 3
due 2023-04-28 at 11:59 PM
Please solve 3 of the 6 problems!

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1 Exercise 1

1.1 Problem

Which of the exercises 1, 2, 3, 4, 5 from homework set #1 remain true if "simple graph" is replaced by "multigraph"?

(For each exercise that becomes false, provide a counterexample. For each exercise that remains true, either provide a new solution that works for multigraphs, or argue that the solution we have seen applies verbatim to multigraphs, or derive the multigraph case from the simple graph case.)

1.2 Solution

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2 Exercise 2

2.1 Problem

Let G be a multigraph. Let $d > 2$ be an integer. Assume that $\deg v > 2$ for each vertex v of G. Prove that G has a cycle whose length is not divisible by d.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let $G = (V, E, \varphi)$ be a multigraph that has no loops.

If $e \in E$ is an edge that contains a vertex $v \in V$, then we let e/v denote the endpoint of e distinct from v. (If e is a loop, then this is understood to mean v itself.)

For each $v \in V$, we define a rational number q_v by

$$q_v = \sum_{\substack{e \in E; \\ v \in \varphi(e)}} \frac{\deg(e/v)}{\deg v}.$$

(Note that the denominator $\deg v$ on the right hand side is nonzero whenever the sum is nonempty!)

(Thus, q_v is the average degree of the neighbors of v, weighted with the number of edges that join v to the respective neighbors. If v has no neighbors, then $q_v = 0$.)

Prove that

$$\sum_{v \in V} q_v \geq \sum_{v \in V} \deg v.$$

(In other words, in a social network, your average friend has, on average, more friends than you do!)

3.2 Hint

Any two positive reals x and y satisfy $\frac{x}{y} + \frac{y}{x} \geq 2$. Why, and how does this help?

3.3 Solution

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4 Exercise 4

4.1 Problem

Let n and k be two integers such that $n > k > 0$. Define the simple graph $Q_{n,k}$ as follows: Its vertices are the bitstrings $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$; two such bitstrings are adjacent if and only if they differ in exactly k bits[1]. (Thus, $Q_{n,1}$ is the n-hypercube graph Q_n.)

(a) Does $Q_{n,k}$ have a hamc[2] when k is even?

(b) Does $Q_{n,k}$ have a hamc when k is odd?

4.2 Remark

One way to approach part **(b)** is by identifying the set $\{0,1\}$ with the field $\mathbb{F}_2$ with two elements. The bitstrings $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ thus become the size-n row vectors in the $\mathbb{F}_2$-vector space $\mathbb{F}_2^n$. Let $e_1, e_2, \ldots, e_n$ be the standard basis vectors of $\mathbb{F}_2^n$ (so that e_i has a 1 in its i-th position and zeroes everywhere else). Then, two vectors are adjacent in the n-hypercube graph Q_n (resp. in the graph $Q_{n,k}$) if and only if their difference is one of the standard basis vectors (resp., a sum of k distinct standard basis vectors). Try to use this to find a graph isomorphism from Q_n to a subgraph of $Q_{n,k}$.

4.3 Solution

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5 Exercise 5

5.1 Problem

Let $k \in \mathbb{N}$. Let S be a finite set. The *Kneser graph* $K_{S,k}$ is the simple graph whose vertices are the k-element subsets of S, and whose edges are the unordered pairs $\{A, B\}$ consisting of two such subsets A and B that satisfy $A \cap B = \varnothing$.

Prove that this Kneser graph $K_{S,k}$ is connected if $|S| \geq 2k + 1$.

5.2 Remark

Can the "if" here be replaced by an "if and only if"? Not quite, because the graph $K_{S,k}$ is also connected if $|S| = 2$ and $k = 1$ (in which case it has two vertices and one edge), or if $|S| = k$ (in which case it has only one vertex). There might be more such "exceptions".

[1] In other words: Two vertices $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ are adjacent if and only if the number of $i \in \{1, 2, \ldots, n\}$ satisfying $a_i \neq b_i$ equals k.

[2] Recall that "hamc" is short for "Hamiltonian cycle".

5.3 Solution

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6 Exercise 6

6.1 Problem

Let $n \geq 1$. Let Q_n be the n-hypercube graph, as in Spring 2022 Lecture 6 (or Spring 2023 Lecture 4).

At what vertices can a hamp[3] of Q_n end if it starts at the vertex $00 \cdots 0$? (Find all possibilities, and prove that they are possible and all other vertices are impossible.)

6.2 Solution

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[3] Recall that "hamp" is short for "Hamiltonian path".