9. Math 235 Fall 2023, Worksheet 9: Assorted problems II

Here comes another smörgåsbord worksheet. No new theory to read up on (although the class problems have been chosen to introduce some mildly new ideas). Use the time to finish reading Worksheet 8 :)

Some of the problems here come from enumerative combinatorics. See [Grinbe20, Chapter 7] for a much more comprehensive introduction to that. (On this worksheet, however, we will need only the simplest few of its ideas, which should be well-known.)

I am also using this as an occasion to introduce some very basic combinatorial game theory. For more, see (e.g.) [KarPer16, Chapter 1] and [AlNoWo19].

As before, $\mathbb{N}$ means the set $\{0, 1, 2, \ldots\}$.

9.1. Class problems

The following problems are to be discussed during class.

Exercise 9.1.1. Let $k \in \mathbb{N}$ and $n \in \mathbb{N}$. Let $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ be an n-tuple of elements of $\{1, 2, \ldots, k\}$.

Prove that there are exactly $k + n(k-1)$ distinct $(n+1)$-tuples that can be obtained from $\mathbf{a}$ by inserting a further element of $\{1, 2, \ldots, k\}$ at some position.

[**Example:** If $k = 3$ and $n = 3$ and $\mathbf{a} = (1, 3, 1)$, then the distinct $(n+1)$-tuples we can obtain this way are

$$(1,1,3,1), \quad (2,1,3,1), \quad (3,1,3,1), \quad (1,2,3,1),$$
$$(1,3,3,1), \quad (1,3,1,1), \quad (1,3,2,1), \quad (1,3,1,2), \quad (1,3,1,3).$$

There are 9 of them. Note that we said "distinct", so we don't count $(1,1,3,1)$ twice even though $(1,1,3,1)$ can be obtained from $(1,3,1)$ in two different ways.]

Exercise 9.1.2. Let n be a positive integer. Two players play the following game: Initially, n candles burn on a table. The two players alternatingly make moves, where each move consists in blowing out an arbitrary number of candles between 1 and 5 inclusive (he chooses this number). The player to blow out the last candle wins.

For what n does the first player (i.e., the player making the first move) have a winning strategy?

For what n does the second player have a winning strategy?

Exercise 9.1.3. Let $n \in \mathbb{N}$. An *n-bitstring* shall mean an n-tuple $(a_1, a_2, \ldots, a_n) \in \{0, 1\}^n$ (that is, an n-tuple of 0's and 1's). Clearly, there are 2^n many n-bitstrings. (For example, $(1, 1, 0, 1)$ is a 4-bitstring.)

(a) An n-bitstring $(a_1, a_2, \ldots, a_n)$ is said to be *lacunar* if it contains no two consecutive 1's (that is, there exists no $i \in \{1, 2, \ldots, n-1\}$ such that $a_i = a_{i+1} = 1$). How many lacunar n-bitstrings are there?

(b) An n-bitstring $(a_1, a_2, \ldots, a_n)$ is said to be *slow* if it contains no entry that differs from both its neighbors (i.e., there exists no $i \in \{2, 3, \ldots, n-1\}$ such that a_i is distinct from both a_{i-1} and a_{i+1}). How many slow n-bitstrings are there?

(c) What is the total number of 1's summed over all n-bitstrings?

(d) What is the total number of 1's summed over all slow n-bitstrings?

Exercise 9.1.4. Consider a circle with center O. Three points on this circle are chosen randomly (each one chosen according to the uniform distribution, independently of the other two). What is the probability that O lies inside the triangle that these three points form?

(Simpler variant of Putnam contest 1992 problem A6)

Exercise 9.1.5. Let A and B be two 2×2-matrices with integer entries. Assume that for each $k \in \{1, 2, 3, 4, 5\}$, the matrix $A + kB$ is invertible and its inverse has integer entries. Prove that this holds for all $k \in \mathbb{Z}$.

[**Hint:** If U is an invertible matrix, then $\det\left(U^{-1}\right) = (\det U)^{-1}$. What can you say about $\det(A + tB)$ as a polynomial in t ?]

9.2. Homework exercises

Solve 4 of the 10 exercises below and upload your solutions on gradescope by December 12.

Exercise 9.2.1. Consider the same game as in Exercise 9.1.2, but with a minor difference: Now, each player is allowed to blow out 1, 2 or 4 candles (rather than $k \in \{1, 2, 3, 4, 5\}$ candles).

(a) Same question as before: Who wins (depending on n), assuming optimal play?

(b) What if, instead, each player can blow out 2 or 3 candles? Blowing out the second-to-last candle also counts as winning here (since it leaves the next player with no possible moves).

(c) Generalize part **(b)**: Fix an integer $m > 2$. Assume that each player can blow out an arbitrary number $k \in \{2, 3, \ldots, m-1\}$ of candles. Who wins?

Exercise 9.2.2. Let A and B be two $n \times n$-matrices with complex entries such that $AB + A + B = 0$. Prove that $BA + B + A = 0$ as well.

[**Hint:** One of the many facets of the infamous Invertible Matrix Theorem says the following: If X and Y are two $n \times n$-matrices that satisfy $XY = I_n$ (where I_n is the $n \times n$ identity matrix), then $YX = I_n$ as well.]

Exercise 9.2.3. Let n be a positive integer. Consider the 2^n many n-bitstrings (see Exercise 9.1.3 for their definition). Assume that 2^{n-1} of these 2^n bitstrings have been colored white (arbitrarily), while the remaining 2^{n-1} have been colored black. Prove that there exist at least 2^{n-1} pairs (w, b), where w is a white n-bitstring, where b is a black n-bitstring, and where w and b differ in exactly one position.

[**Example:** The two 4-bitstrings $(0, 1, 0, 0)$ and $(0, 1, 1, 0)$ differ in exactly one position (the third one).]

Exercise 9.2.4. Let a, b, c be three positive reals satisfying $a^2 + b^2 + c^2 = 1$. Show that $\dfrac{bc}{a} + \dfrac{ca}{b} + \dfrac{ab}{c} \geq \sqrt{3}$.

(BWM Germany 2019, 2nd round, problem 2)

Exercise 9.2.5. Let n be a positive integer such that $n \equiv -1 \bmod 12$. Let s be the sum of all positive divisors of n. Prove that $12 \mid s$.

Exercise 9.2.6. For each positive integer n, let $\operatorname{god}(n)$ denote the greatest odd divisor of n. For instance, $\operatorname{god}(24) = 3$ and $\operatorname{god}(25) = 25$ and $\operatorname{god}(30) = 15$.

Given two positive integers a and b. We define a sequence $(x_1, x_2, x_3, \ldots)$ of positive integers recursively by setting $x_1 = a$ and $x_2 = b$ and

$$x_n = \operatorname{god}(x_{n-1} + x_{n-2}) \qquad \text{for all } n \geq 3.$$

Prove that all entries of this sequence from a certain point on are equal to $\operatorname{god}(\gcd(a, b))$. In other words, prove that there exists a $k \geq 1$ such that $x_k = x_{k+1} = x_{k+2} = \cdots = \operatorname{god}(\gcd(a, b))$.

(Tournament of Towns 30.6.3, by G. Galperin)

Exercise 9.2.7. Let p and q be two coprime positive integers. Let $s = p + q$.

(a) Prove that $\{(ip) \% s \mid i \in \{1, 2, \ldots, s-1\}\} = \{1, 2, \ldots, s-1\}$. (That is, the $s - 1$ remainders $(1p) \% s$, $(2p) \% s$, $\ldots$, $((s-1)p) \% s$ are just the $s - 1$ numbers $1, 2, \ldots, s - 1$ in some order.)

(b) Let $\mathbf{a} = (a_1, a_2, \ldots, a_{s-1})$ be an $(s-1)$-tuple of arbitrary objects (e.g., numbers). Given an integer $d \in \{1, 2, \ldots, s-1\}$, we say that $\mathbf{a}$ is *d-periodic* if each $i \in \{1, 2, \ldots, s-1-d\}$ satisfies $a_i = a_{i+d}$.

Assume that $\mathbf{a}$ is both p-periodic and q-periodic. Prove that $\mathbf{a}$ is 1-periodic (i.e., we have $a_1 = a_2 = \cdots = a_{s-1}$).

(c) Generalize part **(b)**: Let $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ be a k-tuple of arbitrary objects, where $k \geq s-1$. Given an integer $d \in \{1, 2, \ldots, k\}$, we say that $\mathbf{a}$ is *d-periodic* if each $i \in \{1, 2, \ldots, k-d\}$ satisfies $a_i = a_{i+d}$.

Assume that $\mathbf{a}$ is both p-periodic and q-periodic. Prove that $\mathbf{a}$ is 1-periodic (i.e., we have $a_1 = a_2 = \cdots = a_k$).

Exercise 9.2.8. Let $a_1, a_2, \ldots, a_n$ be n integers. A pair (i, j) of integers satisfying $1 \leq i < j \leq n$ will be called

- an *odd inversion* if a_i is odd, a_j is even, and $j - i$ is odd;
- an *even inversion* if a_i is odd, a_j is even, and $j - i$ is even.

Prove that the number of odd inversions is at least as large as the number of even inversions.

(Tournament of Towns 30.6.4, by V. Yassinski)

Exercise 9.2.9. We write the numbers $1, 2, \ldots, n^2$ into the n^2 cells of an $n \times n$-chessboard (where $n > 1$). (Each number goes into exactly one cell.)

Prove that we can find two adjacent cells of this chessboard in which the two numbers differ by more than n. Here, "adjacent" means "have an edge or a vertex in common" (so a cell can have up to 8 other cells adjacent to it).

(Tournament of Towns 12.5, by N. Sedrakyan)

Exercise 9.2.10. Continue with the notations of Exercise 9.1.3. Recall also the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$, defined by $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for each $n \geq 2$.

Prove that the total number of 1's summed over all lacunar n-bitstrings is

$$\frac{nf_{n+1} + 2(n+1) f_n}{5}.$$

References

[AlNoWo19] Michael H. Albert, Richard J. Nowakowski, David Wolfe, *Lessons in Play: An Introduction to Combinatorial Game Theory*, 2nd edition 2019.

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`

[KarPer16] Anna R. Karlin, Yuval Peres, *Game Theory, Alive*, 13 December 2016. `https://homes.cs.washington.edu/~karlin/GameTheoryBook.pdf`