7. Math 235 Fall 2023, Worksheet 7: Assorted problems

This worksheet, for a change, contains no new theory or methods, but just an assortment of various problems I like.

As before, $\mathbb{N}$ means the set $\{0, 1, 2, \ldots\}$.

7.1. Class problems

The following problems are to be discussed during class.

Exercise 7.1.1. Let $n \in \mathbb{N}$. Assume that the set $\{1, 2, \ldots, 2n\}$ is partitioned into two disjoint subsets A and B, each of which has size n.

Let $a_1, a_2, \ldots, a_n$ be the elements of A in increasing order (that is, $a_1 < a_2 < \cdots < a_n$).

Let $b_1, b_2, \ldots, b_n$ be the elements of B in decreasing order (that is, $b_1 > b_2 > \cdots > b_n$).

Prove that

$$|a_1 - b_1| + |a_2 - b_2| + \cdots + |a_n - b_n| = n^2.$$

(Soviet Union 1985 grade 8 problem 8, known as "Proizvolov's identity")

Exercise 7.1.2. Let P be a polynomial with integer coefficients. Let a, b, c be three integers that satisfy $P(a) = b$ and $P(b) = c$ and $P(c) = a$. Prove that $a = b = c$.

Exercise 7.1.3. Consider the sequence $(u_0, u_1, u_2, \ldots)$ of numbers defined recursively by

$$\begin{aligned} u_0 &= 1; \\ u_{2n} &= u_n + u_{n-1} \qquad \text{for all positive } n \in \mathbb{N}; \\ u_{2n+1} &= u_n \qquad \text{for all } n \in \mathbb{N}. \end{aligned}$$

(As we recall, $\mathbb{N} = \{0, 1, 2, \ldots\}$.) Prove that for each positive rational number k, there is a unique $n \in \mathbb{N}$ such that $u_n / u_{n+1} = k$.

(Stronger version of Putnam 2002 Problem A5)

Exercise 7.1.4. Let a, b, c, d be four integers such that $ab = cd$. Prove that there exist four integers x, y, z, w such that

$$a = xy, \qquad b = zw, \qquad c = xz, \qquad d = yw.$$

Exercise 7.1.5. Let $n \in \mathbb{N}$.

(a) Prove that $\left(2+\sqrt{3}\right)^n + \left(2-\sqrt{3}\right)^n$ is an integer.

(b) Prove that $\left\lfloor \left(2+\sqrt{3}\right)^n \right\rfloor$ is odd. (Here, for any given real x, the notation $\lfloor x \rfloor$ stands for the largest integer that is $\leq x$.)

7.2. Homework exercises

Solve 4 of the 10 exercises below and upload your solutions on gradescope by November 29.

Exercise 7.2.1. Let P be a polynomial (in one indeterminate) with integer coefficients. Let n be an odd positive integer. Let $a_1, a_2, \ldots, a_n$ be n integers that satisfy

$$P(a_i) = a_{i+1} \qquad \text{for each } i \in \{1, 2, \ldots, n\},$$

where we set $a_{n+1} := a_1$. Prove that $a_1 = a_2 = \cdots = a_n$.

Exercise 7.2.2. Let $(a_1, a_2, a_3, \ldots)$ be a sequence of reals such that $a_1 = 1$ and such that all positive integers k satisfy

$$1^2 a_1 + 2^2 a_2 + 3^2 a_3 + \cdots + k^2 a_k = \frac{k(k+1)}{2}(a_1 + a_2 + \cdots + a_k).$$

Find an explicit formula for a_n.

Exercise 7.2.3. Let n be a positive integer. Prove that either the number n or the number $3n$ (or both) begins with one of the digits 1, 2 and 9 when written in the decimal system.

(**Example:** If $n = 52$, then $3n = 156$ begins with a 1.)

(Tournament of Towns 26.5.2, deobfuscated version)

Exercise 7.2.4. For each positive integer n, we let $\operatorname{god}(n)$ denote the greatest odd divisor of n. For instance, $\operatorname{god}(24) = 3$ and $\operatorname{god}(25) = 25$ and $\operatorname{god}(30) = 15$.

Let n be a positive integer. Prove that

$$\operatorname{god}(n+1) + \operatorname{god}(n+2) + \cdots + \operatorname{god}(2n) = n^2.$$

(Tournament of Towns 25.1.3)

Exercise 7.2.5. Let $n \in \mathbb{N}$. Let A be an $n \times n$-matrix whose all entries are odd integers.

Consider the product of all entries in each row and each column of A. Prove that the sum of these $2n$ products is congruent to $2n$ modulo 4.

(Variant of Soviet Union 1962 grade 8 problem 5)

Exercise 7.2.6. Let n be a positive integer. Let A be an $n \times n$-matrix whose all entries belong to the set $\{0, 1\}$. Assume that A contains equally many 0's and 1's. Prove that A has two rows that contain the same number of 1's or two columns that contain the same number of 1's. (The "or" is not exclusive, so A can contain both.)

(Tournament of Towns 36.F.H.1)

Exercise 7.2.7. Let $\varphi = \dfrac{1+\sqrt{5}}{2}$ be the golden ratio. Define a function $F : \mathbb{N} \to \mathbb{N}$ by

$$F(n) = \lfloor \varphi n \rfloor \qquad \text{for each } n \in \mathbb{N}.$$

(Recall that $\lfloor x \rfloor$ denotes the largest integer $\leq x$ for each $x \in \mathbb{R}$.)

Recall also the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$, defined by $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for each $n \geq 2$.

(a) Prove that $F(f_n) = f_{n+1}$ for each odd $n \geq 1$.

(b) Prove that $F(f_n) = f_{n+1} - 1$ for each even $n \geq 1$.

(c) Prove that

$$F(F(n)) = F(n) + n - 1 \qquad \text{for each positive integer } n.$$

[**Hint:** Feel free to use the fact that φ is irrational.]

Exercise 7.2.8. Let n be a positive integer such that $4n+1$ is a prime number. Prove that $4n + 1 \mid n^{2n} - 1$.

(Baltic Way contest 2018 problem 18)

Exercise 7.2.9. Let A be an $m \times n$-matrix whose entries are nonnegative reals and which has more columns than rows (that is, we have $n > m$). Assume that each column of A contains at least one positive entry.

Prove that A has at least one cell with a positive entry such that the sum of the entries in the row of this cell is larger than the sum of all entries in the column of this cell.

(BWM Germany 2020, 2nd round, problem 4)

Exercise 7.2.10. Prove that there exists no function $f : \mathbb{Z} \to \mathbb{Z}$ such that each $x \in \mathbb{Z}$ satisfies $f(f(x)) = x + 1$.

[Note the contrast to Exercise 0.3.7 on worksheet #0.]