# 1. Math 235 Fall 2023, Worksheet 1: Induction

This is not an introduction to mathematical induction; I assume you are familiar with the latter already. You can find explanations of the different forms of induction, as well as basic examples, in [LeLeMe16, Chapter 5], [Day16], [Vellem06, Chapter 6], [Hammac15, Chapter 10], [Vorobi02], [Grinbe15, Chapter 2]. We will not discuss why induction is like a domino effect, or why all horses don't have the same color, or how to prove the commutativity of multiplication in $\mathbb{Z}$. Our goal here is rather to explore how induction can be used in non-obvious and advanced ways to solve nontrivial problems. This will be just a little selection from a vast realm; see [AndCri17] or [Gunder10] or [Grinbe20, Chapter 2] for more.

The format of this worksheet is as follows: We begin with a few example problems (to which we give rough outlines of solutions), then a few more problems to be discussed during class, and finally 6 homework problems of which you are supposed to solve 3.

We will abbreviate the word "induction hypothesis" by "IH". The notation $\mathbb{N}$ denotes $\{0, 1, 2, \ldots\}$.

## 1.1. Example problems

### 1.1.1. The Fibonacci sequence

We begin with some properties of the Fibonacci sequence. First, we recall its definition (not least because its numbering is not standardized across the literature):

**Definition 1.1.1.** The *Fibonacci sequence* is the sequence $(f_0, f_1, f_2, \ldots)$ of integers which is defined recursively by

$$f_0 = 0, \qquad f_1 = 1, \qquad \text{and} \qquad f_n = f_{n-1} + f_{n-2} \text{ for all } n \geq 2.$$

The first *Fibonacci numbers* (this is how the entries of the Fibonacci sequence are called) are listed in the following table:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $f_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 |

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Here is a bunch of properties of the Fibonacci sequence, all of which can be proved by simple induction arguments:

**Exercise 1.1.1.** Let $(f_0, f_1, f_2, \ldots)$ be the Fibonacci sequence. Prove that:

**(a)** Each integer $n \geq 0$ satisfies

$$f_1 + f_2 + \cdots + f_n = f_{n+2} - 1.$$

**(b)** Each positive integer $n$ satisfies

$$f_{n+1}f_{n-1} - f_n^2 = (-1)^n.$$

(This is known as *Cassini's identity*.)

**(c)** For any nonnegative integers $n$ and $m$, we have

$$f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}.$$

(This is known as the *addition formula for Fibonacci numbers*.)

**(d)** Let $\varphi = \dfrac{1+\sqrt{5}}{2}$ and $\psi = \dfrac{1-\sqrt{5}}{2}$ be the two solutions of the quadratic equation $X^2 - X - 1 = 0$. Then,

$$f_n = \frac{1}{\sqrt{5}}\varphi^n - \frac{1}{\sqrt{5}}\psi^n \tag{1}$$

for every nonnegative integer $n$. (This is known as *Binet's formula*. The number $\varphi \approx 1.618$ or its inverse $\varphi^{-1} \approx 0.618$ is known as the *golden ratio*.)

**(e)** If $a$ and $b$ are two nonnegative integers satisfying $a \mid b$, then $f_a \mid f_b$.

**(f)** We have $\gcd(f_n, f_{n+1}) = 1$ for each $n \geq 0$. (As usual, $\gcd(u, v)$ denotes the greatest common divisor of two integers $u$ and $v$.)

**(g)** We have $f_2 < f_3 < f_4 < \cdots$.

*Solution idea.* **(a)** (See [Grinbe20, Exercise 2.2.1] for details.) This is a straightforward induction on $n$. The induction base (the case $n = 0$) is trivial, since $f_1 + f_2 + \cdots + f_0$ is an *empty sum* (i.e., a sum with no addends) and therefore equals 0 by definition.[1] The induction step (from $n = k$ to $n = k+1$) follows by observing that[2]

$$\begin{aligned} f_1 + f_2 + \cdots + f_{k+1} &= \underbrace{(f_1 + f_2 + \cdots + f_k)}_{\substack{= f_{k+2} - 1 \\ \text{(by IH)}}} + f_{k+1} = f_{k+2} - 1 + f_{k+1} \\ &= \underbrace{(f_{k+2} + f_{k+1})}_{\substack{= f_{k+3} \\ \text{(by Definition 1.1.1)}}} - 1 = f_{k+3} - 1. \end{aligned}$$

**(b)** (See [Grinbe20, Exercise 2.2.2] for details.) This is again an easy induction on $n$. The induction base (that's the case $n = 1$ here) is trivial. The induction step

[1] Yes, this is part of any good definition of a sum. Likewise, an empty product is defined to be 1.
[2] The abbreviation "IH" means "induction hypothesis".

(from $n = k$ to $n = k + 1$) proceeds by observing that

$$\underbrace{f_{k+2}}_{\substack{=f_{k+1}+f_k \\ \text{(by Definition 1.1.1)}}} f_k - f_{k+1}^2 = (f_{k+1} + f_k) f_k - f_{k+1}^2 = f_k^2 - f_{k+1} \underbrace{(f_{k+1} - f_k)}_{\substack{=f_{k-1} \\ \text{(why?)}}}$$

$$= f_k^2 - f_{k+1} f_{k-1} = - \underbrace{\left(f_{k+1} f_{k-1} - f_k^2\right)}_{\substack{=(-1)^k \\ \text{(by IH)}}} = -(-1)^k = (-1)^{k+1}.$$

**(c)** (See [Grinbe20, Exercise 2.2.3] for details.) This is slightly trickier, as we now have two variables. If nothing else, we have to decide which one to induct on. Here, this is not a real issue since they play symmetric roles, so we can pick either. But note that inducting on $n$ and inducting on $m$ are not the only options; sometimes, it is better to induct on some "derivate quantity" such as $n + m$ (this is actually pretty frequent; we will see such an example below) or $|n - m|$ (this is more exotic). We are in luck: Inducting on $n$ will do. After all, this is an introductory exercise.

There is another subtlety: Do we first fix $m$ and induct on $n$ (that is, having fixed $m$, we use induction to prove the statement "$f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$ for all $n \geq 0$"), or do we induct on $n$ right away (that is, we use induction to prove the statement "($f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$ for all $m \geq 0$) for all $n \geq 0$")? The second option is generally preferable to the first one, since anything we could do with a fixed $m$ could also be done with $m$ being a variable. (The only downside of the second option is that we have to carry the "for all $m \geq 0$" quantifier around.)

Let us thus take the second option. Thus, for any $n \geq 0$, we let $\mathcal{A}(n)$ denote the statement "$f_{n+m+1} = f_n f_m + f_{n+1} f_{m+1}$ for all $m \geq 0$". Our goal is thus to prove that $\mathcal{A}(n)$ holds for all $n \geq 0$. We prove this by induction on $n$. The base case is again trivial (why?). For the induction step, we fix a nonnegtive integer $k$ and assume (as our IH) that $\mathcal{A}(k)$ holds, and we need to prove that $\mathcal{A}(k+1)$ holds. So we need to prove that $f_{(k+1)+m+1} = f_{k+1} f_m + f_{(k+1)+1} f_{m+1}$ for all $m \geq 0$. And this we can do, again, by some mildly strategic computation: For any $m \geq 0$, we have

$$f_{(k+1)+m+1} = f_{k+(m+1)+1} = f_k f_{m+1} + f_{k+1} \underbrace{f_{(m+1)+1}}_{\substack{=f_{m+2}=f_{m+1}+f_m \\ \text{(by Definition 1.1.1)}}}$$

$$\left( \begin{array}{c} \text{here, we have applied } \mathcal{A}(k) \text{, which our IH guarantees} \\ \text{to be true; but we have applied it to } m+1 \text{ instead of } m \\ \text{(which is allowed, since it has a "for all } m \geq 0\text{" quantifier)} \end{array} \right)$$

$$= f_k f_{m+1} + f_{k+1} (f_{m+1} + f_m) = f_{k+1} f_m + \underbrace{(f_{k+1} + f_k)}_{\substack{=f_{k+2} \\ \text{(by Definition 1.1.1)}}} f_{m+1}$$

$$= f_{k+1} f_m + f_{k+2} f_{m+1} = f_{k+1} f_m + f_{(k+1)+1} f_{m+1}.$$

So we are done. Note that we really profited from the fact that we did not fix $m$ here, because otherwise we would not have been able to apply the IH to $m+1$ instead of $m$.

**(d)** (See [Grinbe20, Theorem 2.3.1] for details.)

This requires *strong induction*. See [Grinbe20, §2.3] or [Grinbe15, §2.8] for detailed explanations of this method; here we just summarize how it works: In regular induction, you prove that a statement $\mathcal{A}(n)$ holds for all integers $n \geq 0$ by first showing that $\mathcal{A}(0)$ holds (the "induction base"), and then showing that $\mathcal{A}(k-1) \Longrightarrow \mathcal{A}(k)$ holds for each $k \geq 1$ (the "induction step"). In other words, you prove that a statement $\mathcal{A}(n)$ holds for all integers $n \geq 0$ by first showing that it holds for 0, and then showing that it holds for a positive integer $k$ if it holds for the preceding integer $k-1$. In contrast, in strong induction, you achieve the same goal by showing that

$$(\mathcal{A}(0) \wedge \mathcal{A}(1) \wedge \cdots \wedge \mathcal{A}(k-1)) \Longrightarrow \mathcal{A}(k) \qquad \text{holds for each } k \geq 0 \qquad (2)$$

(the "induction step" of the strong induction). That is, you show that your statement $\mathcal{A}(n)$ holds for $n = k$ under the assumption that it holds for all $n \in \{0, 1, \ldots, k-1\}$ (that is, for all nonnegative integers $n < k$). In other words, you show that the statement holds for a nonnegative integer $k$ if it holds not just for the preceding integer $k-1$, but rather for **all** nonnegative integers smaller than $k$. That is, your IH is not just $\mathcal{A}(k-1)$ but rather $\mathcal{A}(0) \wedge \mathcal{A}(1) \wedge \cdots \wedge \mathcal{A}(k-1)$. Thus the name "strong induction": It is like regular induction but with a stronger IH.

(The attentive reader will have spotted another difference: A strong induction has no induction base! A moderately attentive reader might be puzzled by this: How can we get anywhere without a base case? How do we know that $\mathcal{A}(0)$ holds? Of course, a really attentive reader will see the answer: It now follows from the induction step! Indeed, note that (2) says "$k \geq 0$", not "$k > 0$" (unlike for regular induction). So the induction step in a strong induction has to work for $k = 0$ in particular. However, for $k = 0$, the "induction hypothesis" (that is, the antecedent $\mathcal{A}(0) \wedge \mathcal{A}(1) \wedge \cdots \wedge \mathcal{A}(k-1)$ in (2)) is vacuously true (because it is just saying that $\mathcal{A}(n)$ holds for all nonnegative integers $n < k$; but there are no such integers because $k = 0$), and thus (2) gives us $\mathcal{A}(0)$ unconditionally. So, technically speaking, a strong induction needs no base case, but this is merely because its induction step has to include the $k = 0$ case and the IH is vacuous in this case.)

Enough talking, let us solve Exercise 1.1.1 **(d)** by strong induction. Thus, for any $n \geq 0$, we let $\mathcal{A}(n)$ denote the statement (1). As we said, no base case is required. Here is the induction step: Let $k \geq 0$ be an integer. We must prove (2). Thus, we assume that $\mathcal{A}(0) \wedge \mathcal{A}(1) \wedge \cdots \wedge \mathcal{A}(k-1)$ holds (this is our IH), and we want to prove that $\mathcal{A}(k)$ holds.

If $k = 0$ or $k = 1$, then we can check this by hand. (This is just saying that we manually check that (1) is true for $n = 0$ and for $n = 1$.) So we WLOG assume that $k \geq 2$. Then, our IH yields that $\mathcal{A}(k-1)$ and $\mathcal{A}(k-2)$ hold. (This is where we need $k \geq 2$ ! In fact, if $k$ was $< 2$, then $\mathcal{A}(k-2)$ wouldn't even make sense, since $f_{k-2}$ would not be defined. Thus, in a sense, we didn't cheat our way around needing a base case; we still had to manually verify $\mathcal{A}(0)$ and $\mathcal{A}(1)$. It's just that we did it as part of the induction step, as opposed to calling it a base case.)

So we know that $\mathcal{A}(k-1)$ and $\mathcal{A}(k-2)$ hold. In other words, we have

$$f_{k-1} = \frac{1}{\sqrt{5}}\varphi^{k-1} - \frac{1}{\sqrt{5}}\psi^{k-1} \qquad \text{and} \tag{3}$$

$$f_{k-2} = \frac{1}{\sqrt{5}}\varphi^{k-2} - \frac{1}{\sqrt{5}}\psi^{k-2}. \tag{4}$$

Our goal is to show that $\mathcal{A}(k)$ holds, i.e., that $f_k = \frac{1}{\sqrt{5}}\varphi^k - \frac{1}{\sqrt{5}}\psi^k$. We do it in the straightforward way:

$$\begin{aligned}
f_k &= f_{k-1} + f_{k-2} \qquad \text{(by Definition 1.1.1)} \\
&= \left(\frac{1}{\sqrt{5}}\varphi^{k-1} - \frac{1}{\sqrt{5}}\psi^{k-1}\right) + \left(\frac{1}{\sqrt{5}}\varphi^{k-2} - \frac{1}{\sqrt{5}}\psi^{k-2}\right) \qquad \text{(by (3) and (4))} \\
&= \frac{1}{\sqrt{5}}\left(\varphi^{k-1} + \varphi^{k-2}\right) - \frac{1}{\sqrt{5}}\left(\psi^{k-1} + \psi^{k-2}\right).
\end{aligned}$$

If we can show that

$$\varphi^{k-1} + \varphi^{k-2} = \varphi^k \qquad \text{and} \qquad \psi^{k-1} + \psi^{k-2} = \psi^k, \tag{5}$$

then this will simplify to $\frac{1}{\sqrt{5}}\varphi^k - \frac{1}{\sqrt{5}}\psi^k$, and we will be done (as this will prove $\mathcal{A}(k)$ and thus complete the induction step). Thus, all we need is to prove (5). But this is easy: Since $\varphi$ is a root of $X^2 - X - 1$, we have $\varphi^2 = \varphi + 1$ and thus $\varphi^k = \varphi^2\varphi^{k-2} = (\varphi + 1)\varphi^{k-2} = \varphi^{k-1} + \varphi^{k-2}$. Similarly, $\psi^k = \psi^{k-1} + \psi^{k-2}$. This proves (5), and thus our induction step is complete. Part **(d)** is solved.

**(e)** (See [Grinbe20, Exercise 3.2.2] for details.) Again, we have two variables. Inducting on $a$ is not a good idea, since that would make $f_a$ a "moving target". (When proving a divisibility $u \mid v$ by induction, it is generally better to have $u$ change as little as possible during the induction step, so that the IH does not become useless. It is usually easier to derive $u \mid v$ from $u \mid v'$ than to derive $u \mid v$ from $u' \mid v$.)

So let's induct on $b$. With regular induction, this would be impossible: Since the claim we are proving has an "if $a \mid b$" condition, we cannot hope to get any use out of the IH. After all, if $a \mid b$, then $a \nmid b-1$ (unless $a = 1$), so that the IH is vacuously true and equally useless.

One way to deal with this issue is to use strong induction instead of regular induction. This is done in [Grinbe20, Exercise 3.2.2].

Here, let us take another way. Recall that the claim in question is "if $a$ and $b$ are two nonnegative integers satisfying $a \mid b$, then $f_a \mid f_b$". However, two nonnegative integers $a$ and $b$ satisfy $a \mid b$ if and only if there exists a nonnegative integer $n$ such that $b = an$. This allows us to substitute $an$ for $b$, thus rewriting our claim as "If $a$ and $n$ are nonnegative integers, then $f_a \mid f_{an}$". Then, we prove this rewritten claim by inducting on $n$. The underlying idea here is to replace a condition ("if $a \mid b$") by a parameterization (in our case, $b = an$) that encodes this condition. This is a powerful technique for simplifying problems, and useful not just for induction problems; it is generally fair to say that claims tend to become easier to prove the fewer conditions they have.

Here are the details of the argument. To simplify our life, we fix $a \in \mathbb{N}$. (As explained above, fixing a variable ahead of an induction is generally not a good idea, as it takes some options away; but in our current situation, we can allow ourselves to do it. And if we couldn't, we could always say "oops" and go back and unfix it... Just don't do it on the clean copy!)

Now, for each integer $n \geq 0$, we let $\mathcal{A}(n)$ denote the statement "$f_a \mid f_{an}$". Our goal is thus to prove that this statement $\mathcal{A}(n)$ holds for all $n \geq 0$. We shall do this by induction on $n$. The base case ($n = 0$) is trivial, since $f_{a \cdot 0} = f_0 = 0$ is divisible by every integer[3]. For the induction step, we fix some $k \geq 0$, and we assume (as IH) that $\mathcal{A}(k)$ holds; our goal is to prove $\mathcal{A}(k+1)$.

Our assumption $\mathcal{A}(k)$ is saying that $f_a \mid f_{ak}$. Our goal ($\mathcal{A}(k+1)$) is saying that $f_a \mid f_{a(k+1)}$. How do we get the latter from the former? Since multiplication is repeated addition, it makes sense to apply the addition formula (Exercise 1.1.1 **(c)**). We want to apply it in such a way that $a(k+1)$ will be $n + m + 1$, but the right hand side will divisible by $f_a$. Given these premises, a little bit of experimentation tells us what to do: We apply Exercise 1.1.1 **(c)** to $n = ak$ and $m = a - 1$, and obtain

$$f_{ak+a-1+1} = \underbrace{f_{ak} f_{a-1}}_{\substack{\text{This is divisible by } f_a, \\ \text{since } f_a \mid f_{ak}}} + \underbrace{f_{ak+1} f_{a-1+1}}_{\substack{\text{This is divisible by } f_a, \\ \text{because } f_{a-1+1} = f_a}} .$$

Thus, $f_a \mid f_{ak+a-1+1} = f_{a(k+1)}$ (since $ak + a - 1 + 1 = a(k+1)$). This proves $\mathcal{A}(k+1)$, thus completing our induction step.

**(f)** We induct on $n$. The base case ($n = 0$) boils down to $\gcd(0,1) = 1$, which is clear. For the induction step (from $k-1$ to $k$), we assume that some $k \geq 1$ satisfies $\gcd(f_{k-1}, f_k) = 1$, and we set out to show that $\gcd(f_k, f_{k+1}) = 1$. (By the way, it is perfectly fine to use the letter $n$ instead of $k$ here; just don't write awkward things like "the claim is true for $n = n - 1$".)

We will need the following two basic properties of gcds:

1. We have $\gcd(a,b) = \gcd(b,a)$ for any $a, b \in \mathbb{Z}$.

2. We have $\gcd(a, ua + b) = \gcd(a,b)$ for any $a, b, u \in \mathbb{Z}$.

[3]Alas, some authors like to claim that 0 is not divisible by 0, because $\frac{0}{0}$ is undefined. It is, of course, their right to use their favorite definition of divisibility, but we prefer to use the more reasonable "$u \mid v$ holds if and only if there exists an integer $w$ such that $v = uw$" definition.

Both of these properties are fundamental to the Euclidean algorithm, as they say that the gcd of two integers does not change when we swap the two integers or add a multiple of one to the other. They easily follow from the definition of the gcd.[4]

Now, $f_{k+1} = f_k + f_{k-1} = 1 f_k + f_{k-1}$. Hence,

$$\begin{aligned} \gcd(f_k, f_{k+1}) &= \gcd(f_k, 1 f_k + f_{k-1}) = \gcd(f_k, f_{k-1}) \\ &\qquad\qquad (\text{by Property 2, applied to } a = f_k,\ b = f_{k-1} \text{ and } u = 1) \\ &= \gcd(f_{k-1}, f_k) \qquad (\text{by Property 1}) \\ &= 1 \qquad (\text{by the IH}). \end{aligned}$$

This completes the induction step, and thus Exercise 1.1.1 **(f)** is solved.

**(g)** It is easy to see that the integers $f_1, f_2, f_3, \ldots$ are positive[5]. Thus, every $i \geq 2$ satisfies $f_{i-1} > 0$ and therefore $f_i < f_{i+1}$ (because $f_{i+1} = f_i + \underbrace{f_{i-1}}_{>0} > f_i$). In other words, $f_2 < f_3 < f_4 < \cdots$. This solves Exercise 1.1.1 **(g)**. □

We now come to a slightly deeper property of Fibonacci numbers. We claim that every nonnegative integer can be uniquely represented as a sum of non-consecutive Fibonacci numbers $f_n$ with $n \geq 2$ (where "non-consecutive" means that $f_k$ and $f_{k+1}$ can never appear together in the sum). For example,

$$50 = \underbrace{34}_{=f_9} + \underbrace{13}_{=f_7} + \underbrace{3}_{=f_4} = f_9 + f_7 + f_4.$$

This fact is known as *Zeckendorf's theorem*. Let us state it more formally:

**Exercise 1.1.2.** A set $S$ of integers will be called *lacunar* if it contains no two consecutive integers (i.e., if there exists no $s \in S$ such that $s + 1 \in S$).

Let $(f_0, f_1, f_2, \ldots)$ be the Fibonacci sequence.

For any $n \in \mathbb{N}$, there exists a unique finite lacunar subset $S$ of $\{2, 3, 4, \ldots\}$ such that $n = \sum_{s \in S} f_s$.

*Solution idea.* (See [Grinbe18, Theorem 2.4] for some details.) We will soon prove this by induction. But first, we shall state a lemma and prove it (by induction, too). Of course, when you solve a problem, you won't magically come up with the lemma ahead of the solution most of the time; instead, it will emerge (as a statement you wish to be true) in the process of solving the problem. The reason why we are starting with the lemma here is to uncrowd the solution that will follow it. We

[4]To be precise, property 1 is obvious, while property 2 follows from the following easy lemma:

*Lemma.* Let $a, b, u \in \mathbb{Z}$. Let $d$ be a divisor of $a$. Then, $d$ divides $ua + b$ if and only if $d$ divides $b$.

[5]If you insist, this requires another proof by induction, albeit a trivial one.

could put it in the middle of the solution, but this would be needlessly confusing and complicated (as a rule of style, an induction within an induction is almost always to be avoided). So let me state the lemma first; you can trust me that it will be exactly what we will need later on:

> **Lemma 1.1.2.** Let $u \in \mathbb{N}$. Let $S$ be a lacunar subset of $\{2, 3, \ldots, u\}$. Then, $\sum\limits_{s \in S} f_s < f_{u+1}$.

This lemma gives an upper bound on the sum of a set of non-consecutive Fibonacci numbers in terms of the largest Fibonacci number that appears in this set (or, more precisely, of the next Fibonacci number that appears after it). The proof is by induction:

*Proof of Lemma 1.1.2.* We proceed by strong induction on $u$.

*Induction step:* Let $v \in \mathbb{N}$. Assume (as the IH) that Lemma 1.1.2 holds for all $u < v$. We must prove that Lemma 1.1.2 holds for $u = v$. In other words, we must prove that if $S$ is a lacunar subset of $\{2, 3, \ldots, v\}$, then $\sum\limits_{s \in S} f_s < f_{v+1}$.

So let $S$ be a lacunar subset of $\{2, 3, \ldots, v\}$. We must show that $\sum\limits_{s \in S} f_s < f_{v+1}$. If $S$ is empty, then this is clear (since $\sum\limits_{s \in S} f_s = 0$ in this case, but $f_{v+1} > 0$). Hence, we WLOG assume that $S$ is nonempty. Thus, $S$ has a largest element, which we shall call $m$. Since $S$ is a subset of $\{2, 3, \ldots, v\}$, we have $m \leq v$. Since $S$ is lacunar, the second-largest element of $S$ (if it exists) is not only $< m$, but also $< m - 1$; in other words, the set $S \setminus \{m\}$ is a subset of $\{2, 3, \ldots, m - 2\}$. Of course, this set $S \setminus \{m\}$ is furthermore lacunar.

Now, our IH tells us that Lemma 1.1.2 holds for all $u < v$. Hence, in particular, Lemma 1.1.2 holds for $u = m - 2$ (because $m - 2 < m \leq v$). Thus, we can apply Lemma 1.1.2 to $m - 2$ and $S \setminus \{m\}$ instead of $u$ and $S$ (since $S \setminus \{m\}$ is a lacunar subset of $\{2, 3, \ldots, m - 2\}$), and thus obtain $\sum\limits_{s \in S \setminus \{m\}} f_s < f_{(m-2)+1} = f_{m-1}$.

However, $m \in S$, so that

$$\sum_{s \in S} f_s = f_m + \underbrace{\sum_{s \in S \setminus \{m\}} f_s}_{< f_{m-1}} < f_m + f_{m-1} = f_{m+1} \leq f_{v+1}$$

(since $m \leq v$ entails $m + 1 \leq v + 1$ and therefore $f_{m+1} \leq f_{v+1}$ by Exercise 1.1.1 **(g)**). This inequality is precisely the one we wanted to show. Thus, we have proved that Lemma 1.1.2 holds for $u = v$. The induction step is complete, and with it the proof of Lemma 1.1.2. $\square$

Now, let us solve the actual Exercise 1.1.2.

We apply strong induction on $n$. For the induction step, we fix an integer $k \geq 0$, and we assume (as the IH) that Exercise 1.1.2 holds for each $n < k$. We must now prove that Exercise 1.1.2 holds for $n = k$.

If $k = 0$, then this is trivial (the only subset $S$ of $\{2,3,4,\ldots\}$ satisfying $0 = \sum_{s\in S} f_s$ is the empty set, since any other subset $S$ would lead to $\sum_{s\in S} f_s > 0$). Thus, we WLOG assume that $k > 0$.

Exercise 1.1.1 **(g)** yields $f_2 < f_3 < f_4 < \cdots$. Thus, the sequence $(f_2, f_3, f_4, \ldots)$ grows unboundedly. Hence, there exists a **largest** $i \in \{2,3,4,\ldots\}$ satisfying $f_i \leq k$ (why?). Consider this $i$. Then, $f_i \leq k < f_{i+1}$.

From $k < f_{i+1}$, we obtain $k - f_i < f_{i+1} - f_i = f_{i-1}$ (why?). This will be useful later on.

Now, $k - f_i$ is a nonnegative integer (since $f_i \leq k$) and is $< k$ (since $f_i > 0$). Hence, the IH shows that Exercise 1.1.2 holds for $n = k - f_i$. Thus, there exists a unique finite lacunar subset $S'$ of $\{2,3,4,\ldots\}$ such that

$$k - f_i = \sum_{s\in S'} f_s. \tag{6}$$

Consider this $S'$.

We next claim that each element of $S'$ is smaller than $i-1$. Indeed, if $S'$ would contain some $j \geq i-1$, then we would have

$$\sum_{s\in S'} f_s \geq f_j \geq f_{i-1} \qquad \left(\text{since } j \geq i-1, \text{ thus } f_j \geq f_{i-1} \text{ (why?)}\right),$$

which would contradict $\sum_{s\in S'} f_s = k - f_i < f_{i-1}$. Thus, each element of $S'$ is smaller than $i-1$. In particular, $S'$ contains neither $i-1$ nor $i+1$. Hence, the set $S' \cup \{i\}$ is still lacunar (since $S'$ is lacunar). Moreover, from (6), we obtain $k = f_i + \sum_{s\in S'} f_s = \sum_{s\in S'\cup\{i\}} f_s$ (since $i \notin S'$ (why?)).

Thus, we have found a finite lacunar subset $S$ of $\{2,3,4,\ldots\}$ such that $k = \sum_{s\in S} f_s$ (namely, $S = S' \cup \{i\}$). This means we are halfway to our goal; we still need to show that this subset $S$ is unique.

In order to show this, we fix some finite lacunar subset $S$ of $\{2,3,4,\ldots\}$ such that $k = \sum_{s\in S} f_s$. Our goal is to show that this $S$ must be our previously constructed $S' \cup \{i\}$. Indeed, this will clearly entail the uniqueness of $S$, and thus complete the induction step.

We will first show that $i \in S$. It is here that Lemma 1.1.2 reveals its usefulness. Indeed, the set $S$ is nonempty (since $\sum_{s\in S} f_s = k > 0$), so it has a largest element. Call this largest element $m$. If we had $m \geq i+1$, then we would have

$$\sum_{s\in S} f_s \geq f_m \geq f_{i+1} \qquad \left(\text{by Exercise 1.1.1 } \textbf{(g)}, \text{ since } m \geq i+1\right),$$

which would contradict $\sum_{s\in S} f_s = k < f_{i+1}$. Thus, we must have $m < i+1$. On the other hand, if we had $m \leq i-1$, then $S$ would be a subset of $\{2,3,\ldots,i-1\}$, and therefore Lemma 1.1.2 (applied to $u = i - 1$) would yield $\sum_{s \in S} f_s < f_{(i-1)+1} = f_i \leq k$, which would contradict $\sum_{s \in S} f_s = k$. Hence, we must have $m > i - 1$. Combining this with $m < i + 1$, we obtain $m = i$ (since $m$ and $i$ are integers). Thus, $i = m \in S$. Now,

$$k = \sum_{s \in S} f_s = f_i + \sum_{s \in S \setminus \{i\}} f_s \qquad \text{(since } i \in S\text{)},$$

so that

$$k - f_i = \sum_{s \in S \setminus \{i\}} f_s.$$

Thus, $S \setminus \{i\}$ is a finite lacunar subset of $\{2, 3, 4, \ldots\}$ such that $k - f_i = \sum_{s \in S \setminus \{i\}} f_s$. Hence, $S \setminus \{i\} = S'$ (because $S'$ is the **unique** finite lacunar subset $S'$ of $\{2, 3, 4, \ldots\}$ such that $k - f_i = \sum_{s \in S'} f_s$, but we just saw that $S \setminus \{i\}$ is another such subset). Therefore, $S = S' \cup \{i\}$ (since $i \in S$). This is precisely what was left to be proved. Thus, the induction step is complete. This solves Exercise 1.1.2. $\square$

### 1.1.2. More general recursions

As we have seen, the Fibonacci numbers make for a nice playground for induction proofs, but their real significance is as one of the simplest examples of what I'd call a $(u, v)$-recurrent sequence (which in turn is a particular case of many more comprehensive concepts):

**Definition 1.1.3.** Let $u$ and $v$ be two numbers. A sequence $(x_0, x_1, x_2, \ldots)$ of numbers will be called *$(u, v)$-recurrent* if every $n \geq 2$ satisfies

$$x_n = u x_{n-1} + v x_{n-2}. \tag{7}$$

It is clear that an $(u, v)$-recurrent sequence $(x_0, x_1, x_2, \ldots)$ is uniquely determined by the four numbers $u$, $v$, $x_0$ and $x_1$, since the equality (7) can be used to compute all entries of the sequence using these four numbers. The Fibonacci sequence $(f_0, f_1, f_2, \ldots)$ is the $(1, 1)$-recurrent sequence with the starting entries $f_0 = 0$ and $f_1 = 1$. Other examples of $(u, v)$-recurrent sequences are arithmetic progressions (these are precisely the $(2, -1)$-recurrent sequences) and the geometric progressions (these are $(u, 0)$-recurrent) as well as the sequences of the forms

$$(\sin(\alpha + 0\beta), \sin(\alpha + 1\beta), \sin(\alpha + 2\beta), \ldots) \qquad \text{and}$$
$$(\cos(\alpha + 0\beta), \cos(\alpha + 1\beta), \cos(\alpha + 2\beta), \ldots)$$

for any angles $\alpha$ and $\beta$.

Many properties of Fibonacci numbers (e.g., Exercise 1.1.1, but not Exercise 1.1.2) can be generalized to arbitrary (or at least fairly general) $(u, v)$-recurrent sequences. Here is what becomes of Exercise 1.1.1 when we generalize it:

**Exercise 1.1.3.** Let $u$ and $v$ be two numbers (e.g., real or complex numbers). Let $(x_0, x_1, x_2, \ldots)$ be any $(u, v)$-recurrent sequence of numbers. Prove that:

**(a)** If $v = 1$, then each integer $n \geq 0$ satisfies

$$u\left(x_1 + x_2 + \cdots + x_n\right) = x_{n+1} + x_n - x_1 - x_0.$$

**(b)** Each positive integer $n$ satisfies

$$x_{n+1}x_{n-1} - x_n^2 = (-v)^{n-1}\left(x_2x_0 - x_1^2\right).$$

**(c)** For any nonnegative integers $n$ and $m$, we have

$$vx_0x_{n+m} + x_1x_{n+m+1} = vx_nx_m + x_{n+1}x_{m+1}. \tag{8}$$

More generally, if $(y_0, y_1, y_2, \ldots)$ is a further $(u, v)$-recurrent sequence, then

$$vx_0y_{n+m} + x_1y_{n+m+1} = vx_ny_m + x_{n+1}y_{m+1}. \tag{9}$$

**(d)** Let $\lambda = \dfrac{u + \sqrt{u^2 + 4v}}{2}$ and $\mu = \dfrac{u - \sqrt{u^2 + 4v}}{2}$ be the two solutions of the quadratic equation $X^2 - uX - v = 0$. If $u^2 + 4v \neq 0$ (that is, $\lambda \neq \mu$), then we have

$$x_n = \gamma\lambda^n + \delta\mu^n \tag{10}$$

for every nonnegative integer $n$, where

$$\gamma = \frac{x_1 - \mu x_0}{\lambda - \mu} \qquad \text{and} \qquad \delta = \frac{\lambda x_0 - x_1}{\lambda - \mu}.$$

On the other hand, if $u^2 + 4v = 0$ (that is, $\lambda = \mu$), then we have

$$x_n = \frac{1}{2^n}\left(2nu^{n-1}x_1 - (n-1)\,u^n x_0\right) \tag{11}$$

(where we agree to understand $nu^{n-1}$ as 0 when $n = 0$).

**(e)** Assume that $u$, $v$ and all of $x_0, x_1, x_2, \ldots$ are integers. Assume furthermore that $x_0 = 0$. If $a$ and $b$ are two nonnegative integers satisfying $a \mid b$, then $x_a \mid x_b$.

**(f)** Assume that $u$, $v$ and all of $x_0, x_1, x_2, \ldots$ are integers. Assume furthermore that $x_1 = 1$ and $\gcd(u, v) = 1$. Then, we have $\gcd(x_n, x_{n+1}) = 1$ for each $n \geq 0$. (As usual, $\gcd(u, v)$ denotes the greatest common divisor of two integers $u$ and $v$.)

**(g)** If $x_0 \geq 0$ and $x_1 > 0$ and $u \geq 1$ and $v > 0$, then we have $x_2 < x_3 < x_4 < \cdots$.

*Solution idea.* Almost all parts of this exercise can be solved in the same way as the corresponding parts of Exercise 1.1.1 (which are just their particular cases for $u = 1$ and $v = 1$ and $x_i = f_i$). Try it! The only complications arise in parts **(d)**, **(e)** and **(f)**.

The complication in part **(d)** stems from the case $\lambda = \mu$, which requires some care but is still equally straightforward as the other case. (See [Grinbe20, Theorem 4.9.11] for details.)

In part **(e)**, we observe that (8) becomes $x_1 x_{n+m+1} = v x_n x_m + x_{n+1} x_{m+1}$ (since $x_0 = 0$). Proceeding as in Exercise 1.1.1 **(e)**, we can try to apply this equality to $n = ak$ and $m = a - 1$ in the hope of concluding that $x_a \mid x_{a(k+1)}$ (assuming that $x_a \mid x_{ak}$). Unfortunately, the $x_1$ factor on the left hand side of this equality prevents this argument from working: we only get $x_a \mid x_1 x_{a(k+1)}$, not $x_a \mid x_{a(k+1)}$.

However, an easy trick gets you past this factor: If $x_1 = 0$, then all entries of the sequence $(x_0, x_1, x_2, \ldots)$ are 0 (why?), so the claim holds obviously. Thus, WLOG assume that $x_1 \neq 0$. Now, consider the sequence $\left(\frac{x_0}{x_1}, \frac{x_1}{x_1}, \frac{x_2}{x_1}, \ldots\right)$. This sequence is $(u, v)$-recurrent (why?). The first entries of this sequence are $\frac{x_0}{x_1} = \frac{0}{x_1} = 0$ and $\frac{x_1}{x_1} = 1$; thus, all entries of this sequence are integers (why?). So we can WLOG assume that $x_1 = 1$ (otherwise, just replace our original sequence $(x_0, x_1, x_2, \ldots)$ by $\left(\frac{x_0}{x_1}, \frac{x_1}{x_1}, \frac{x_2}{x_1}, \ldots\right)$). Now, the $x_1$ factor is not in our way any more; (8) becomes $x_{n+m+1} = v x_n x_m + x_{n+1} x_{m+1}$, which is perfectly sufficient for our induction step. (See [Grinbe20, Exercise 4.10.2] for details.)

The complication in part **(f)** is probably the most interesting one, as it requires a new idea to overcome. Again, we induct on $n$ (as in Exercise 1.1.1 **(f)**). The base case ($n = 0$) boils down to $\gcd(x_0, 1) = 1$, which is clear. For the induction step (from $k - 1$ to $k$), we assume that some $k \geq 1$ satisfies $\gcd(x_{k-1}, x_k) = 1$, and we set out to show that $\gcd(x_k, x_{k+1}) = 1$. From $x_{k+1} = u x_k + v x_{k-1}$, we obtain[6]

$$\begin{aligned} \gcd(x_k, x_{k+1}) &= \gcd(x_k, u x_k + v x_{k-1}) = \gcd(x_k, v x_{k-1}) \\ &\qquad\qquad (\text{by Property 2, applied to } a = x_k \text{ and } b = x_{k-1}) \\ &= \gcd(v x_{k-1}, x_k) \qquad (\text{by Property 1}). \end{aligned}$$

What now? Our IH tells us that $\gcd(x_{k-1}, x_k) = 1$, not that $\gcd(v x_{k-1}, x_k) = 1$. It looks like we are stuck: Our IH is too weak to yield the goal. In a sense, we did not pack enough provisions into our backpack.

The trick is to pack more. That is, we start from scratch and try again, but this time we try to prove a stronger claim. Namely, instead of proving that $\gcd(x_n, x_{n+1}) = 1$ for each $n \geq 0$, we try to prove that

$$\gcd(v, x_{n+1}) = 1 \qquad \text{and} \qquad \gcd(x_n, x_{n+1}) = 1 \tag{12}$$

[6]We are using the Properties 1 and 2 that were stated in our above solution to Exercise 1.1.1 **(f)**.

for each $n \geq 0$. The new statement "$\gcd(v, x_{n+1}) = 1$" here is the extra provision we are packing into our backpack in order to have it handy when we need it in the induction step.

So let us try again. The base case is still trivial (since $x_1 = 1$). For the induction step (from $k-1$ to $k$), we assume that some $k \geq 1$ satisfies

$$\gcd(v, x_k) = 1 \qquad \text{and} \qquad \gcd(x_{k-1}, x_k) = 1,$$

and we set out to show that

$$\gcd(v, x_{k+1}) = 1 \qquad \text{and} \qquad \gcd(x_k, x_{k+1}) = 1.$$

As before, we can show that $\gcd(x_k, x_{k+1}) = \gcd(vx_{k-1}, x_k)$. However, this time we know that both $\gcd(v, x_k) = 1$ and $\gcd(x_{k-1}, x_k) = 1$; this easily yields that $\gcd(vx_{k-1}, x_k) = 1$. Indeed, we have the following property of gcds:

3. If $a, b, c$ are three integers satisfying $\gcd(a, c) = 1$ and $\gcd(b, c) = 1$, then $\gcd(ab, c) = 1$.

This is not as simple as properties 1 and 2, but is still easily proved once you know some basic number theory (Bezout's identity or the prime factorization theorem). See, e.g., [Grinbe20, Theorem 3.5.10] for a proof. (Keep in mind that [Grinbe20] uses the notation "$p \perp q$" for "$\gcd(p, q) = 1$".)

Anyway, Property 3 lets us derive $\gcd(vx_{k-1}, x_k) = 1$ from $\gcd(v, x_k) = 1$ and $\gcd(x_{k-1}, x_k) = 1$. Because of $\gcd(x_k, x_{k+1}) = \gcd(vx_{k-1}, x_k)$, this rewrites as $\gcd(x_k, x_{k+1}) = 1$.

This is half the battle. The other half is proving that $\gcd(v, x_{k+1}) = 1$. We again recall that $x_{k+1} = ux_k + vx_{k-1} = vx_{k-1} + ux_k$, and thus

$$\begin{aligned}
&\gcd(v, x_{k+1}) \\
&= \gcd(v, vx_{k-1} + ux_k) = \gcd(v, ux_k) \qquad \text{(again by Property 2 (why?))} \\
&= \gcd(ux_k, v) \\
&= 1 \qquad \text{(by Property 3, since } \gcd(u, v) = 1 \text{ and } \gcd(x_k, v) = \gcd(v, x_k) = 1\text{)}.
\end{aligned}$$

So we are done with the induction step, and Exercise 1.1.3 **(f)** is solved. (See [Grinbe20, Exercise 4.9.8] for details.) □

The above solution to Exercise 1.1.3 **(f)** illustrates one of the major intricacies in the use of induction: Often, the exact claim that one wants to prove cannot be directly proved by induction, and paradoxically the trick is often to **strengthen** the claim (aka pack extra provisions into the "induction backpack"). Of course, any such strengthening is a tradeoff: You have more to prove, but you also have more to use in the induction step. Finding the right strengthening is an art, not a science, and it is here where the most interesting ideas often lie.

### 1.1.3. Induction in combinatorics

Here is another unusual type of induction argument:

> **Exercise 1.1.4.** We will say that a finite sequence $(x_1, x_2, \ldots, x_n)$ of integers is *nice* if there exist no three integers $i, j, k$ with
>
> $$1 \leq i < j < k \leq n \qquad \text{and} \qquad x_i + x_k = 2x_j.$$
>
> (In other words, a finite sequence is nice if and only if it contains no three-term arithmetic progression as a subsequence. Or, to put it slightly differently: It is nice if and only if no two of its entries have their average written anywhere between them.)
>
> Prove that for every integer $n \geq 0$, there exists a nice permutation of the sequence $(1, 2, \ldots, n)$.

For example, a nice permutation of the sequence $(1, 2, 3, 4, 5, 6, 7)$ is $(1, 5, 3, 7, 2, 6, 4)$. On the other hand, there are plenty of non-nice permutations of $(1, 2, 3, 4, 5, 6, 7)$; for example, $(4, 1, 3, 7, 8, 2, 5, 6)$ is not nice (since the two entries 1 and 5 have their average – that is, 3 – written between them). Of course, $(1, 2, 3, 4, 5, 6, 7)$ itself is not nice (and is, in fact, as far from "nice" as a permutation can be).

*Solution idea.* We let $\mathcal{A}(n)$ denote the statement "there exists a nice permutation of the sequence $(1, 2, \ldots, n)$". Our goal is thus to prove $\mathcal{A}(n)$ for all $n \geq 0$.

We could try regular induction, i.e., we could try to prove that $\mathcal{A}(k) \Longrightarrow \mathcal{A}(k+1)$ for each $k \geq 0$. However, it is not immediately clear how to do this.

However, it is easy to show that $\mathcal{A}(k) \Longrightarrow \mathcal{A}(2k)$ for each $k \geq 0$. In other words, if we have an integer $k \geq 0$ and we know that there exists a nice permutation of $(1, 2, \ldots, k)$, then we can easily construct a nice permutation of $(1, 2, \ldots, 2k)$. To wit, if $(a_1, a_2, \ldots, a_k)$ is our given nice permutation of $(1, 2, \ldots, k)$, then the sequence

$$(2a_1,\ 2a_2,\ \ldots,\ 2a_k,\ 2a_1 - 1,\ 2a_2 - 1,\ \ldots,\ 2a_k - 1)$$

is a nice permutation of $(1, 2, \ldots, 2k)$. (Check this! The trick is that the average between an even and an odd integer is not an integer; thus, if two entries of this sequence had their average written anywhere between them, then these two entries would be either both among the first $k$ entries or among the last $k$ entries. But in each of these cases, we would easily get a contradiction to the niceness of $(a_1, a_2, \ldots, a_k)$.)

So we have shown that $\mathcal{A}(k) \Longrightarrow \mathcal{A}(2k)$ for each $k \geq 0$. Since we can also easily check that $\mathcal{A}(1)$ holds (indeed, the trivial permutation $(1)$ of $(1)$ is nice, since it has no two entries to begin with), we thus obtain $\mathcal{A}(1) \Longrightarrow \mathcal{A}(2) \Longrightarrow \mathcal{A}(4) \Longrightarrow \mathcal{A}(8) \Longrightarrow \cdots$. Thus, $\mathcal{A}(n)$ is proved whenever $n$ is a power of 2. In other words,

$$\mathcal{A}(2^m) \text{ holds for each } m \geq 0. \tag{13}$$

However, it is also easy to show that $\mathcal{A}(k) \Longrightarrow \mathcal{A}(k-1)$ for each $k \geq 1$. Indeed, if we have an integer $k \geq 1$ and we know that there exists a nice permutation of $(1, 2, \ldots, k)$, then we can easily obtain a nice permutation of $(1, 2, \ldots, k-1)$. (To wit, we simply remove the entry $k$ from our nice permutation.)

Using this "backward induction step" $\mathcal{A}(k) \Longrightarrow \mathcal{A}(k-1)$ multiple times, we conclude that we have

$$\mathcal{A}(u) \Longrightarrow \mathcal{A}(v) \text{ whenever } u \in \mathbb{N} \text{ and } v \in \mathbb{N} \text{ satisfy } u \geq v. \tag{14}$$

Now, using (13) and (14), we can easily see that $\mathcal{A}(n)$ holds for every $n \geq 0$. Indeed, let us fix some $n \geq 0$. Then, there exists some $m \in \mathbb{N}$ such that $2^m \geq n$ (because the powers of 2 are unbounded). Consider this $m$. Now, (13) tells us that $\mathcal{A}(2^m)$ holds. Then, (14) lets us conclude that $\mathcal{A}(n)$ holds (since $2^m \geq n$). Thus, $\mathcal{A}(n)$ is proven for every $n \geq 0$. This solves Exercise 1.1.4. $\square$

This strange version of induction that we just used – with two induction steps, one being $\mathcal{A}(k) \Longrightarrow \mathcal{A}(2k)$ and the other being $\mathcal{A}(k) \Longrightarrow \mathcal{A}(k-1)$ – is known as *Cauchy induction*.

Finally, here is a tricky combinatorial exercise ([Grinbe20, Exercise 3.7.9]):

**Exercise 1.1.5.** Let $p, q, m, n \in \mathbb{N}$ with $p \leq m$ and $q \leq n$. Consider an $m \times n$-table $T$ of integers, with all entries distinct. In each **c**olumn of $T$, we mark the $p$ largest entries with a ***c**yan* marker. In each **r**ow of $T$, we mark the $q$ largest entries with a ***r**ed* marker. Prove that at least $pq$ entries of $T$ are marked twice (i.e., with both colors).

[**Example:** Let $p = 2$ and $q = 2$ and $m = 3$ and $n = 3$ and

$$T = \begin{pmatrix} 1 & 2 & 9 \\ 4 & 3 & 8 \\ 5 & 6 & 7 \end{pmatrix}.$$

Then,

$$\begin{aligned} &\text{the cyan entries are } 4, 5, 3, 6, 8, 9, \text{ while} \\ &\text{the red entries are } 2, 9, 4, 8, 6, 7. \end{aligned}$$

Thus, the entries $4, 6, 8, 9$ are marked twice. This is exactly the $pq$ entries claimed in the exercise. You can easily find situations in which there are more than $pq$ doubly-marked entries.]

*Solution idea.* (See [Grinbe20, §A.2.9] for details.) We induct on $m + n$. The base case ($m + n = 0$) is trivial (since $m + n = 0$ entails $m = n = 0$ and thus $p = q = 0$).

*Induction step:* Let $k \in \mathbb{N}$. Assume (as the IH) that Exercise 1.1.5 holds for $m + n = k$. We must prove that Exercise 1.1.5 holds for $m + n = k + 1$.

So let $p, q, m, n \in \mathbb{N}$ be such that $p \leq m$ and $q \leq n$ and $m + n = k + 1$. Let $T$ be an $m \times n$-table of integers, with all entries distinct. Mark some of the entries in $T$ with a cyan marker and some with a red marker, as described in the statement of the exercise. We must show that at least $pq$ entries of $T$ are marked twice (i.e., with both colors).

We shall use the abbreviations "cyan", "red", "1-marked" and "2-marked" for "marked cyan", "marked red", "marked with exactly one color" and "marked with both colors", respectively. Thus, our goal is to show that at least $pq$ entries of $T$ are 2-marked.

If $T$ has no 1-marked entries, then this is true (why?). Hence, we WLOG assume that $T$ has some 1-marked entries. Let $M$ be the **largest** 1-marked entry of $T$. We WLOG assume that $M$ is marked cyan (indeed, if it is marked red, then we rotate our table by $90^\circ$, which turns rows into columns and red into cyan and vice versa). Thus, $M$ is marked cyan but not red. Therefore, $p \geq 1$ (why?), so that $p - 1 \in \mathbb{N}$.

Let $R$ be the row of $T$ in which this entry $M$ is located. Then, the $q$ largest entries of the row $R$ are 2-marked in $T$ (why?[7]).

Let us now **remove** the row $R$ from the $m \times n$-table $T$. The result is an $(m-1) \times n$-table $T'$.

We do not copy the cyan and red markings from $T$ to $T'$, but instead we mark some of the entries in $T'$ as follows: In each column of $T'$, we mark the $p - 1$ largest entries with a *cyan* marker. In each row of $T'$, we mark the $q$ largest entries with a *red* marker. It is clear that the red entries of $T'$ are precisely the red entries of $T$ that happen to lie in $T'$ (that is, that are not in row $R$). It is also easy to see that all cyan entries of $T'$ are cyan in $T$ as well (why?), although the converse is not necessarily true[8]. Thus, all entries of $T'$ that are 2-marked in $T'$ must also be 2-marked in $T$.

However, $T'$ is an $(m-1) \times n$-table, and we have $p - 1 \leq m - 1$ and $q \leq n$. Since $(m-1) + n = \underbrace{m+n}_{=k+1} - 1 = k$, we can thus apply the IH to this $(m-1) \times n$-table $T'$ (and to $p - 1$ instead of $p$), and conclude that at least $(p-1)\,q$ entries are 2-marked in $T'$. All of these entries must be 2-marked in $T$ as well (since all entries of $T'$ that are 2-marked in $T'$ must also be 2-marked in $T$). However, the $q$ largest entries of the row $R$ are also 2-marked in $T$ (as we have seen above). Thus, we have found a total of $(p-1)\,q + q$ entries that are 2-marked in $T$. That is, we have found $pq$ entries that are 2-marked in $T$. This completes the induction step. $\square$

## 1.2. Class problems

The following problems are to be discussed during class.

First, some comments on binomial coefficients. We recall that a binomial coeffi-

[7] Hint: They are all red. What would go wrong if any of them was 1-marked?

[8] There might be cyan entries of $T$ that are not cyan in $T'$ (and not just because they are in row $R$).

cient $\dbinom{n}{k}$ is defined by the formula

$$\binom{n}{k} := \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \tag{15}$$

whenever $n$ is any number (integer, real or even complex) and $k$ is a nonnegative integer. When $n \in \mathbb{N}$ and $n \geq k$, there is an equivalent formula

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}; \tag{16}$$

however, if $n \notin \mathbb{N}$ or $n < k$, then the formula (16) becomes meaningless while the formula (15) continues to hold.

**Exercise 1.2.1.** Prove that every $n \in \mathbb{N}$ and $q \in \mathbb{Q}$ satisfy

$$\sum_{r=0}^{n} \binom{r+q}{r} = \binom{n+q+1}{n}.$$

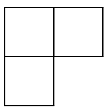
**Exercise 1.2.2.** Let $n$ be a positive integer. Show that

$$\sum_{\substack{(x,y)\in\{1,2,\ldots,n\}^2; \\ \gcd(x,y)=1; \\ x+y>n}} \frac{1}{xy} = 1.$$

**Exercise 1.2.3.** Let $x$ and $y$ be two complex numbers such that $x+y$ and $xy$ are integers. Prove that $x^n + y^n$ is an integer for each $n \in \mathbb{N}$.

**Exercise 1.2.4.** Let $n \in \mathbb{N}$. Consider a $2^n \times 2^n$-chessboard, consisting of $(2^n)^2$ unit squares. A rook is standing on one squares of the chessboard. Prove that it is possible to cover all the remaining squares (i.e., all the $(2^n)^2 - 1$ squares that the rook is not standing on) with L-trominoes in such a way that no two L-trominos overlap.

Here, an *L-tromino* means a shape consisting of three unit squares, looking as follows:

 (possibly rotated).

**Exercise 1.2.5.** We shall consider matrices with real entries. A *saddle entry* of such a matrix means an entry that is the largest entry in its row and the smallest entry in its column. (Ties are allowed – i.e., "the largest" doesn't mean "the only largest".)

Consider an $n \times m$-matrix $A$ (with $n > 0$ and $m > 0$) such that any $2 \times 2$-submatrix of $A$ has a saddle entry. Prove that $A$ has a saddle entry, too.

(A $2 \times 2$-*submatrix* of $A$ is a $2 \times 2$-matrix obtained by picking any two distinct rows of $A$ (not necessarily consecutive) and any two distinct columns of $A$ (not necessarily consecutive), and intersecting these two rows with these two columns. Thus, an $n \times m$-matrix has $\binom{n}{2} \cdot \binom{m}{2}$ many $2 \times 2$-submatrices.)

## 1.3. Homework exercises

Solve 4 of the 10 exercises below and upload your solutions on gradescope by October 10 before class.

**Exercise 1.3.1.** The *Lucas sequence* is the sequence $(\ell_0, \ell_1, \ell_2, \ldots)$ of integers which is defined recursively by

$$\ell_0 = 2, \qquad \ell_1 = 1, \qquad \text{and} \qquad \ell_n = \ell_{n-1} + \ell_{n-2} \text{ for all } n \geq 2.$$

(Thus, this sequence satisfies the same recursive equation as the Fibonacci sequence, differing only in the starting value $\ell_0 \neq f_0$. For instance, $\ell_2 = 3$ and $\ell_4 = 4$ and $\ell_5 = 7$.)

- **(a)** Prove that $\ell_n = f_{n-1} + f_{n+1}$ for each $n \geq 1$.
- **(b)** Prove that $\ell_n^2 - 5f_n^2 = 4 \cdot (-1)^n$ for each $n \geq 0$.

**Exercise 1.3.2.** Let $n$ be a positive integer. For each $k \in \{1, 2, \ldots, n-1\}$, we let

$$a_k := (n-k) \prod_{i=0}^{k-2} (n-i) = (n-k)(n-k+2)(n-k+3) \cdots n.$$

Prove that

$$\sum_{k=1}^{n-1} a_k = n! - 1.$$

[**Hint:** For convenience, rename $a_k$ as $a_{n,k}$ to stress the dependence on $n$.]

The next two exercises are again devoted to binomial coefficients. Some of their basic properties might be required to solve them; see [Grinbe15, §3.1] or [Grinbe19, §1.3] for an overview of these properties (you can use them all without proof).

**Exercise 1.3.3.** Let $n \in \mathbb{N}$. Prove that

$$\sum_{r=1}^{n} \frac{1}{r} \binom{n}{r} = \sum_{r=1}^{n} \frac{2^r - 1}{r}.$$

**Exercise 1.3.4.** Let $n$ and $p$ be positive integers such that $p \leq 2n$. Prove that

$$\sum_{k=p}^{n} 2^k k \binom{2n-k-1}{n-1} = 2^p n \binom{2n-p}{n}.$$

[**Hint:** If you induct on $p$, the base case will be hard. Try to induct on $n - p$ instead! (If $p > n$, then both sides are 0, so you can assume $n - p$ to be nonnegative.)]

You can induct on a nonnegative or positive integer. A slightly modified form of induction works for integers in general (here you need an induction step "from $n$ to $n+1$" as well as a second induction step "from $n$ to $n-1$"). It is not possible to induct on a rational or real number, however. Nevertheless, induction can still be helpful for questions about rational numbers, since a rational number is a ratio of two integers. Try this on the following exercise:

**Exercise 1.3.5.** Let $S$ be a set of nonnegative rational numbers. Assume the following.

1. We have $0 \in S$.

2. For any $x \in S$, we have $\dfrac{1}{x+1} \in S$ and $\dfrac{x}{x+1} \in S$.

Prove that the set $S$ contains all rational numbers in the interval $[0, 1]$.

Next come some exercises of a combinatorial flavor.

**Exercise 1.3.6.** We will say that a finite sequence $(x_1, x_2, \ldots, x_n)$ of integers is *rude* if it there exist no three integers $i, j, k$ with

$$1 \leq i < j < k \leq n \qquad \text{and} \qquad \left(x_i + x_j = 2x_k \text{ or } x_j + x_k = 2x_i\right).$$

(In other words, a finite sequence is rude if and only if no two of its entries have their average written anywhere to the left of them both or to the right of them both.)

Prove that for every integer $n \geq 2$, there exist exactly two rude permutations of the sequence $(1, 2, \ldots, n)$, namely $(1, 2, \ldots, n)$ itself and its reversal $(n, n-1, \ldots, 1)$.

**Exercise 1.3.7.** Let $n$ be a positive integer. In a tournament, $2^{n-1}$ contestants participate, with each pair of (distinct) contestants playing exactly one round against each other ("round-robin tournament"). Each round is won by exactly one player (there are no ties).

Prove that we can find $n$ distinct contestants $c_1, c_2, \ldots, c_n$ such that for each $i < j$, the contestant $c_i$ wins against $c_j$.

**Exercise 1.3.8.** A country has $n$ towns (with $n \geq 1$), arranged along a linear road running from left to right. Each town has a *left bulldozer* (standing on the road to the left of the town and facing left) and a *right bulldozer* (standing on the road to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

For any two towns $P$ and $Q$, we say that town $P$ *dominates* town $Q$ if the bulldozer of $P$ that is facing in the direction of $Q$ can move over to $Q$ without getting pushed off the road.

Prove that there is exactly one town that is not dominated by any other town.

[**Example:** Here is one possibility for $n = 5$:

$$\underline{\quad 7A3 \quad 5B4 \quad 1C9 \quad 2D6 \quad 8E10 \quad},$$

where $A, B, C, D, E$ are the five towns and where each number stands for the size of the corresponding bulldozer. It is easy to check that in this configuration, town $A$ dominates no other town; town $B$ dominates towns $A$, $C$ and $D$; town $C$ dominates $D$ and $E$; town $D$ dominates no other towns; town $E$ dominates town $D$. Thus, the unique undominated town is $B$.]

[**Hint:** What happens if we remove the town with the largest bulldozer?]

**Exercise 1.3.9.** Let $k$ and $n$ be two nonnegative integers. Let $S$ be a set with size $|S| \geq k(n+1) - 1$. Assume that each $n$-element subset of $S$ is colored either red or green. Prove that there exist $k$ pairwise disjoint $n$-element subsets of $S$ that have the same color.

**Exercise 1.3.10.** Let $n$ be a positive integer. You have $n$ boxes, and each box contains a non-negative number of pebbles. In each move, you are allowed to take two pebbles from an arbitrary box, throw away one of the pebbles and put the other pebble in another box. (You can freely choose these two boxes.) An initial configuration of pebbles is called *solvable* if it is possible to reach a configuration with no empty box, in a finite (possibly zero) number of moves.

Prove that a configuration $(a_1, a_2, \ldots, a_n)$ (that is, a configuration in which box

1 has $a_1$ pebbles, box 2 has $a_2$ pebbles, and so on) is solvable if and only if

$$\left\lceil \frac{a_1}{2} \right\rceil + \left\lceil \frac{a_2}{2} \right\rceil + \cdots + \left\lceil \frac{a_n}{2} \right\rceil \geq n.$$

Here, $\lceil x \rceil$ denotes the *ceiling* of a real number $x$ (that is, the smallest integer that is $\geq x$).

[**Example:** The configuration $(3, 0, 6, 0, 0)$ is solvable. Indeed, we can take two pebbles from the first box and move one of them to the second, obtaining $(1, 1, 6, 0, 0)$; then we can take two pebbles from the third box and move one of them to the fourth, obtaining $(1, 1, 4, 1, 0)$; and finally take two pebbles from the third box and move one of them to the fifth, obtaining $(1, 1, 2, 1, 1)$.]

[**Hint:** For a configuration $(a_1, a_2, \ldots, a_n)$, define its *size* to be $a_1 + a_2 + \cdots + a_n$ (that is, the total number of pebbles), and define its *level* to be $\left\lceil \frac{a_1}{2} \right\rceil + \left\lceil \frac{a_2}{2} \right\rceil + \cdots + \left\lceil \frac{a_n}{2} \right\rceil$. Note that the size decreases by 1 with each move, whereas the level either stays the same or decreases by 1 (why?). This suggests making moves that don't decrease the level unless absolutely necessary. When is it necessary?]

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