0. Math 235 Fall 2023, Worksheet 0: Introduction to problem-solving

Welcome to Math 235: a course on mathematical problem-solving. The main "practical" purpose of this course is to prepare for the *Putnam contest*, a 6-hour problem-solving competition that happens annually on the first Saturday of December (such as December 2 in 2023). Its full name is "William Lowell Putnam Mathematical Competition", but everyone calls it the "Putnam contest" or just "the Putnam".

0.0.1. A short history of mathematical competitions

Mathematical problems have been used as puzzles since antiquity; Diophantus' gravestone supposedly contains a mathematical riddle. In the 8th Century AD, Alcuin of York (a scholar at the court of Charlemagne) collected over 50 mathematical puzzles in a book named *Propositiones ad Acuendos Juvenes* ("Problems to Sharpen the Young"). Many of these puzzles have become straightforward with the methods of modern high-school mathematics, but some still remain nontrivial.

Somewhere in the 19th Century, some journals introduced problem sections, in which authors would challenge readers to solve problems until a given deadline, after which the best solutions would get published in the journal (along with a list of all solvers). This tradition still lives on in journals such as *Crux Mathematicorum*, the *American Mathematical Monthly*, the *College Mathematics Journal* or the *Mathematics Magazine*; I recommend you check out the recent problem sections in these journals.

The first *mathematical olympiad* in the modern sense – i.e., exam-like contest with non-straightforward problems – was held in Hungary in 1894. Since then, the Hungarian Mathematics Olympiad for high-school students ("Eötvös Competition") has been administered yearly, and over the 20th Century, its format has been copied by dozens of other countries. Students from all over the world compete in the *International Mathematical Olympiad*; similar international contests have sprung up (e.g.) in Europe and Asia.

While the USA has been late in adopting high-school mathematical olympiads, it can boast one of the first university-level olympiads: the very *Putnam contest* that this course is meant to train you for. It has been held yearly since 1938 (with a pause for WWII), and is open to any undergraduate at a North American university that participates in it (Drexel does). While its results are often used to compare universities, it is actually an individual competition, and every contestant gets scored independently. Among mathematical competitions, it is notorious for the time pressure involved: There are only 6 hours available for 12 problems, each of which can range from nontrivial (the level of a hard homework exercise) to highly challenging (the level of a main result of a low-tier research paper).[1] The median score at the Putnam contest is often 0 or 1 points out of 120 (each problem is worth

[1] In contrast, the International Mathematical Olympiad would give you 4.5 hours for 3 problems.

10 points when correctly solved), and the maximum score is often far below 120; some problems even don't get solved by any participant.[2]

0.0.2. Why play the Putnam?

Why should you partake in such a hopeless and overtaxing competition?

The obvious first answer is "you should if you like a challenge", but there are other reasons as well. You get to play around with ideas in a tenser environment than usual, which can liberate some creativity that would not reveal itself when you sit on the sofa[3]. The potential reward is high, but the risks are low (all you can lose is a few hours of time). Solving a single problem out of 12 gives you bragging rights[4]. Every once in a while, someone gets famous for inventing a particularly surprising or elegant solution.

Finally, problem-solving is research writ small. Even as research projects happen on the scale of months rather than hours, training for the Putnam contest will also build many of the skills needed in research.

Also, there is free lunch between the two parts of the exam (which are conveniently timed 9:00–12:00 AM and 2:00–5:00 PM).

0.0.3. Can problem-solving be learned?

Problem-solving is a vast and open-ended skillset. There is no way to master it; you can only get better and better at it. However, many of the most useful skills are the easiest to pick up. This course will familiarize you with some of the most frequently used techniques, as well as some topics that are not commonly covered in other classes. It does not replace any course on linear algebra, analysis, abstract algebra, combinatorics or probability theory (five mainstays of the Putnam contest), but it covers some minor theories off the beaten path.

We will learn problem-solving by solving problems and discussing the solutions. In particular, the problems of old Putnam contests are a good source of training problems, but we cast our net wider and also use other contests, journals, etc.. No two contests are completely alike, and in particular the Putnam has a certain set of pecularities[5], so you will want to familiarize yourself with the last few decades of

[2]See Kiran Kedlaya's *Putnam archive* (`https://kskedlaya.org/putnam-archive/`) for the problems, solutions and score distributions of the Putnam in recent years.

I never participated in the Putnam myself, as I attended college in Germany, but at my current level of training I would typically score 40–50 points on the exam if I took part.

[3]The opposite effect might also happen. It is best to try out both environments from time to time.

[4]Some employers will count your Putnam score (if it is, say, ≥ 10 or ≥ 20) in your favor, but no one will count its lack against you, since the Putnam is known to be challenging and not everyone's idea of a good time.

[5]Compared to other contests, it is characterized by its aggressive time limit, as well as the presence of "university mathematics" (analysis, probability, some basic abstract algebra) and the near-absence of elementary geometry.

Putnam[6]. Good sources for this are the books [KePoVa01] and [KKKO20] as well as Kiran Kedlaya's problem archive (`https://kskedlaya.org/putnam-archive/`). Also, Gelca's and Andreescu's book [GelAnd17] is a training resource specifically tailored at the Putnam contest and is highly recommended.

This year, the main focus of this course is *problem-solving strategies*, i.e., heuristics and techniques for solving nontrivial problems. In some other years, I teach a different version of this course, which focuses more on *knowledge*, specifically on some of the "mini-theories" that frequently come useful in competitions but are not part of the typical undergraduate curriculum. Notes for the latter version can be found at [Grinbe20]. There is significant overlap between the two versions (some things are too important to miss), but the notes are different enough to be worth a look. A bunch of other texts are cited in [Grinbe20], but let me recommend a few more here:

- David Galvin's notes `https://www3.nd.edu/~dgalvin1/43900/43900_F20/index.html` (click the Overleaf link and compile `main.tex`);
- Richard Stanley's peculiar book [Stanle20] (a good, if distracting, bedtime read);
- the Velleman–Wagon puzzle collections [KoVeWa96] and [WagVel20];
- Goucher's notes [Gouche12] with their unconventional choice of topics;
- Polya's classic [Polya81] with its detailed study of thinking processes and basic steps of problem-solving.

0.0.4. Prerequisites; knowledge vs. lateral thinking

Unlike homework, the typical contest problem does not become straightforward just because you have learned the right topic. In fact, the ideal contest problem is not supposed to reward expertise overly much; it is supposed to test ingenuity and creativity, not voracious learning of fashionable subjects. In practice, however, the typical contest problem rewards both ingenuity and subject knowledge. The two can often be traded for one another, and even when a problem can be solved using the former alone, the latter still tends to be useful, e.g., by putting the right tools in one's hand or revealing certain approaches as hopeless. It rarely hurts to know more.

What kind of knowledge can come useful in the Putnam contest? Theoretically, any kind, but some kinds are more frequently useful.

To **understand** the problems at a Putnam, it is necessary to know the standard undergraduate curriculum (analysis and linear algebra) and high-school mathematics. Occasionally, problems rely on some probability theory and abstract algebra (elementary number theory, groups, rings, fields on the level of a first course).

[6] Older problems (prior to ca. 1980, I guess) are less useful, as the focus of the contest has shifted over time.

All of this can be learned from undergraduate classes (albeit sometimes topics classes) and won't be the focus of this course (although I can give some references at least on the algebraic and combinatorial side).

The tougher question is what can be useful to **solve** the problems at a Putnam. Here is a list of various topics that have been helpful over the last 5 years (2018–2022):

- elementary modular arithmetic $7\times$
- generating functions $7\times$
- degree-2 linear recurrences (Fibonacci sequence and its ilk, Chebyshev polynomials) $4\times$
- basic combinatorics (permutations, combinations, counting, finite probabilities, etc.) $4\times$
- complex roots of polynomials (Fundamental Theorem of Algebra, Gauss–Lucas, etc.) $4\times$
- basic inequalities (AM–GM, Cauchy–Schwarz, rearrangement, Jensen, etc.) $3\times$
- deeper modular arithmetic (quadratic residues, primitive roots etc.) $2\times$
- elementary combinatorial game theory (winning and losing positions) $2\times$
- determinantal identities beyond the basics (e.g., Vandermonde, Cauchy–Binet, adjugate, etc.) $2\times$
- triangle geometry $2\times$
- Fourier transform and series $2\times$
- geometric (i.e., non-discrete) probability $2\times$
- symmetric polynomials (e.g. Newton identities) $1\times$
- advanced linear algebra (e.g., positive definiteness, orthogonal matrices) $1\times$
- differential equations $1\times$
- complex analysis (maximum principle, Cauchy integral, etc.) $1\times$
- Lagrange multipliers $1\times$
- elementary group theory $1\times$
- binomial coefficient manipulations $1\times$
- trigonometric identities $1\times$

(I have made this table by skimming Kedlaya's solutions for the 60 problems of the Putnam contests 2018–2022. The symbol "$k\times$" means that k out of the 60 problems have solutions using the respective topic. Expect some bias, since Kedlaya likes some topics more than others. Note that many problems have several different solutions, and often some of them use advanced techniques while others are elementary. Many problems can be solved with basic knowledge, but become much easier with advanced theories.)

0.0.5. The format of our weekly worksheets

This worksheet (and most future worksheets) consists of

- a reading section (with a few example problems that are discussed in detail) to be read at home,
- a discussion section (with some more problems) to be thought about briefly and then discussed in class, and
- a homework section (with some more problems) to be attempted at home.

Solving a nontrivial problem requires both ingenuity and luck, so I don't expect you to solve everything. For example, this worksheet has 10 homework problems, but you are only expected to solve 5. (But try them all!)

0.1. Example problems

We begin with comparably easy problems. On the Putnam contest, the problems in each 3-hour session are sorted by difficulty, so the easiest problems are A1 and B1 (that is, the first problems of each session), the next-easiest ones are A2 and B2, and so on until A6 and B6. At least this is true in theory. More often than not, the problem committee has a strange idea of what is easy and what is hard, and your mileage will differ significantly. More importantly, a medium-difficulty problem on your favorite subject will often be easier (for you!) than an easy problem on a topic you have never studied. There are quite a few A1/B1 problems that I have never figured out on my own; there are also A6/B6 problems I solved without much trouble. You should expect surprises like this as well. Don't waste the whole exam struggling with an A1; think about the other problems too!

0.1.1. Experiment and discover

The simplest problem-solving strategy of all is to *experiment and discover*[7]. This takes many forms and pervades all sciences; it is the only way to discover something really novel. Nevertheless, it is often forgotten and disused. In concrete situations, "experiment and discover" can mean any of the following:

- If you need to study a sequence of numbers, you compute the first few entries of this sequence (as many as you can comfortably compute) and try to spot patterns in these entries. Then you can try to prove these patterns.
- If you have a problem (say) about the number 500, you first try to solve the analogous problem for the number 3 or the number 5 or (better) for all small numbers for which you can solve it. Then you analyze the results and the ideas used in these small cases, and try to transfer them back to the case of 500.

[7] sometimes abbreviated "FAFO"

- Modify ("mutate") the problem to make it simpler, and see what you can say about the resulting modified problem(s). For instance, if your problem involves a product, you can replace this product by a sum. If your problem is about three-dimensional geometry, you can try to state and solve a two-dimensional analogue. There is no guarantee that solving the modified problem will help solve the original one, but it often serves as a manageable first step.

- If you must solve a system of equations in (say) three variables x, y, z, you can first try to solve it under extra requirements such as $x = 0$ or $x = 1$ or $x = y$. This rarely finds you all solutions, but often gives you the first few.

The following exercises provide some examples for these.

Exercise 0.1.1. Let $(a_0, a_1, a_2, \ldots)$ be a sequence of real numbers defined recursively by $a_0 = 4$ and $a_1 = 1$ and

$$a_n = \frac{1 + a_{n-1}}{a_{n-2}} \qquad \text{for all } n \geq 2.$$

Find a_{1000}.

Solution idea. This is clearly a finite problem (all we need is to compute $a_0, a_1, a_2, \ldots, a_{1000}$ in order, using the recursion given). A computer could do this in a second. But we are not computers and don't have any available at an exam. So we need to find a trick, e.g., a pattern in the sequence that makes it easier to compute.

To find a pattern, we compute the first few entries of our sequence:

n	0	1	2	3	4	5	6	7	8
a_n	4	1	$\frac{1}{2}$	$\frac{3}{2}$	5	4	1	$\frac{1}{2}$	$\frac{3}{2}$

.

The pattern should now be clear: The sequence is 5-periodic, i.e., repeats itself every 5 entries. In other words, $a_{n+5} = a_n$ for each $n \geq 0$. Once noticed, this is straightforward to prove: If we set $u := a_n$ and $v := a_{n+1}$ (for a given $n \geq 0$), then our recursion yields

$$\begin{aligned}
a_{n+2} &= \frac{1+a_{n+1}}{a_n} = \frac{1+v}{u};\\
a_{n+3} &= \frac{1+a_{n+2}}{a_{n+1}} = \frac{1+\dfrac{1+v}{u}}{v} = \frac{1+u+v}{uv};\\
a_{n+4} &= \frac{1+a_{n+3}}{a_{n+2}} = \frac{1+\dfrac{1+u+v}{uv}}{\left(\dfrac{1+v}{u}\right)} = \frac{1+u+v+uv}{v\left(1+v\right)};\\
a_{n+5} &= \frac{1+a_{n+4}}{a_{n+3}} = \frac{1+\dfrac{1+u+v+uv}{v\left(1+v\right)}}{\left(\dfrac{1+u+v}{uv}\right)} = \frac{\left(1+u+v+uv+v\left(1+v\right)\right)u}{\left(1+u+v\right)\left(1+v\right)}\\
&= \frac{\left(1+u+2v+uv+v^2\right)u}{1+u+2v+uv+v^2} \qquad \text{(by expanding the products)}\\
&= u \qquad \text{(by cancellation)}\\
&= a_n,
\end{aligned}$$

which is exactly what we wanted[8]. (Note that we did not use the starting values $a_0 = 4$ and $a_1 = 1$ in this computation; thus, the 5-periodicity $a_{n+5} = a_n$ holds for **any** choice of starting values a_0 and a_1, as long as the sequence is well-defined[9]. This phenomenon is known as the *Lyness 5-cycle*.)

Anyway, this was great but let us not forget to solve the exercise. It asks us to compute a_{1000}. The 5-periodicity of our sequence yields that for every $n \geq 0$, we have

$$a_n = a_{n-5} = a_{n-10} = a_{n-15} = \cdots = a_{n-5q},$$

where $5q$ is the largest (integer) multiple of 5 that can be subtracted from n without going negative. Of course, the $n - 5q$ in this equality is nothing other than the remainder of n upon division by 5; thus, we obtain

$$a_n = a_{n\%5} \qquad \text{for every } n \geq 0,$$

where $n\%5$ is the remainder of n upon division by 5. Applying this to $n = 1000$,

[8] You can simplify the above computation by observing that the numerator $1+u+v+uv$ of a_{n+4} can be factored as $(1+u)(1+v)$, so that $1+v$ can be cancelled from the fraction and we obtain $a_{n+4} = \dfrac{1+u}{v}$. But I deliberately avoided this shortcut to stress that no ingenuity was needed in our proof.

[9] How could the sequence not be well-defined? Well, the recursion $a_n = \dfrac{1+a_{n-1}}{a_{n-2}}$ requires dividing by a_{n-2}, and this fails if $a_{n-2} = 0$. This happens, e.g., if we pick our starting values to be $a_0 = 1$ and $a_2 = -1$ (in which case a_4 will fail to exist, and thus the sequence won't be well-defined).

we find

$$\begin{aligned} a_{1000} &= a_{1000\%5} = a_0 \qquad (\text{since } 1000\%5 = 0) \\ &= 4. \end{aligned}$$

□

Exercise 0.1.2. The *Triscal triangle* is an infinite triangle-shaped table of integers, which looks as follows:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & 1 & 1 & & & \\ & & 1 & 2 & 3 & 2 & 1 & & \\ & 1 & 3 & 6 & 7 & 6 & 3 & 1 & \\ 1 & 4 & 10 & 16 & 19 & 16 & 10 & 4 & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

(going on forever in the western, eastern and southern directions). It is defined as follows: At its apex lies the single integer 1. Each further entry is defined to be the sum of the three closest entries of the row above it (i.e., of the entry immediately north of it, the entry immediately northwest of it, and the immediately northeast of it). (If one of these three entries is not present, we count it as a 0.) For example, the fourth entry of the fifth row is $3 + 6 + 7 = 16$, whereas the third entry of the fourth row is $1 + 2 + 3 = 6$.

The first two rows of the triangle consist entirely of odd entries. Does any other row of this triangle have this property?

(`https://math.stackexchange.com/questions/2693839`)

Solution idea. First, some context. You might be familiar with *the Pascal triangle*, which is a triangular table of integers in which every entry (except for the 1 at the apex) is the sum of the two closest entries in the row above it. This triangle (or, rather, its first 6 rows) looks as follows:

$$\begin{array}{ccccccccccc} & & & & & 1 & & & & & \\ & & & & 1 & & 1 & & & & \\ & & & 1 & & 2 & & 1 & & & \\ & & 1 & & 3 & & 3 & & 1 & & \\ & 1 & & 4 & & 6 & & 4 & & 1 & \\ 1 & & 5 & & 10 & & 10 & & 5 & & 1 \end{array} .$$

It is thus a simpler analogue of the Triscal triangle. It is famous for its myriad properties, including the explicit formula $\dbinom{n}{k} = \dfrac{n!}{k!\,(n-k)!}$ for its entries.

Unfortunately, none of this is helpful for the problem at hand. There is no explicit formula for the entries of the Triscal triangle[10]. Also, the claim of the exercise would be false for the Pascal triangle, as its 1-st, 2-nd, 4-th, 8-th and 16-th rows (for example) consist entirely of odd numbers. (Do you see the pattern in this sequence? Try to prove it; it's another nice problem!)

Anyway, this is a good example of a useless analogy; the Triscal triangle is genuinely different from the Pascal one in this regard. Following this thread just wastes your time (and the more you know about the Pascal triangle, the more time you waste).

So let us start from scratch and experiment directly on the Triscal triangle. We can easily compute some further rows of it, but we can make this even easier with a simple trick: We don't need the entries themselves; we only need to know their parities. (The *parity* of an integer m is the information whether m is even or odd.) To do so, we write "e" for an even integer and "o" for an odd integer. Then, we have $o + o = e$ (by which we mean that the sum of two odd integers is always even) and $o + e = o$ and $e + o = o$ and $e + e = e$. Thus, in order to find out the parity of an entry, we only need to know the parities of the three entries above it, but we don't need the entries themselves. This lets us easily fill in several rows of the Triscal triangle:

$$\begin{array}{ccccccccccccccc}
&&&&&&& o &&&&&&& \\
&&&&&& o & o & o &&&&&& \\
&&&&& o & e & o & e & o &&&&& \\
&&&& o & o & e & o & e & o & o &&&& \\
&&& o & e & e & e & o & e & e & e & o &&& \\
&& o & o & o & e & o & o & o & e & o & o & o && \\
& o & e & o & e & e & e & o & e & e & e & o & e & o & \\
o & o & e & o & o & e & o & o & o & e & o & o & e & o & o
\end{array}.$$

You may see some patterns already, and prove the one or the other quite easily[11]. But let us focus on the problem. As we recall, we need to see whether some row of this triangle (besides the first two) consists entirely of odd entries. The first two rows do, but it appears that they are the only ones: In each further row, we can easily find an even entry (i.e., an "e"). Namely, the 3-rd, 5-th and 7-th rows[12] begin with[13] oe (so that their second entries are already even), whereas the 4-th and 8-th rows begin with ooe, and the sixth row begins with $oooe$. In either case, we find an e (that is, an even entry) among the first four entries of the row; thus, we never need

[10] The simplest way to express the $(k+1)$-st entry in the $(n+1)$-st row of the Triscal triangle is $\sum_{j=0}^{k} \binom{n}{j} \binom{j}{k-j}$. This is too complicated to be helpful.

[11] For example, you should see that the middle entry of each row is odd. Do you see why?

[12] From a mathematically erudite point of view, it would be better to count the rows from 0 (thus placing the apex in the 0-th row instead of the 1-st row). But it does not help in this particular problem, so let us not change horses mid-ride.

[13] "Begin with oe" means that the first two entries of these rows (from west to east) are o and e.

to go beyond the fourth entry in a row. Hence, we can drop these later entries, and obtain the following picture:

$$\begin{array}{ccccccccc}
 & & & & & & & o & \\
 & & & & & & o & o & o \\
 & & & & & o & e & o & e \\
 & & & & o & o & e & o & \\
 & & & o & e & e & e & & \\
 & & o & o & o & e & & & \\
 & o & e & o & e & & & & \\
o & o & e & o & & & & &
\end{array}.$$

Now we see a periodicity pattern: The sequence of the first four entries in a row repeats every 4 rows! Specifically, we observe that for each positive integer k,

- the $(4k)$-th row begins with *ooeo*;
- the $(4k+1)$-th row begins with *oeee*;
- the $(4k+2)$-th row begins with *oooe*;
- the $(4k+3)$-th row begins with *oeoe*.

(You might have to compute a few more rows to spot this pattern, but computing them is easy.)

Having discovered this pattern, we can prove it quickly: If the first four entries of a certain row are a, b, c, d, then the first entries of the next row are $a, a+b, a+b+c, b+c+d$ (by the construction of the triangle). In particular, the latter four entries are fully determined by the former four. Therefore,

- if a row begins with *ooeo*, then the next row begins with *oeee* (since $o = o$, $o+o = e$, $o+o+e = e$ and $o+e+o = e$);
- if a row begins with *oeee*, then the next row begins with *oooe* (for similar reasons);
- if a row begins with *oooe*, then the next row begins with *oeoe* (for similar reasons);
- if a row begins with *oeoe*, then the next row begins with *ooeo* (for similar reasons).

Since the 3-rd row of our triangle begins with *oeoe*, we thus conclude (by applying the above four bullet points) that the 4-th begins with *ooeo*, the 5-th begins with *oeee*, the 6-th begins with *oooe*, the 7-th begins again with *oeoe*, and so on periodically.

In particular, each of these rows will have an "e" (that is, an even entry) among its first four entries. Therefore, it cannot consist entirely of odd entries. We thus have shown that the Triscal triangle has no row that consists entirely of odd entries, except for its first two rows. □

The above two exercises show the power of periodicity patterns. Not every pattern is a periodicity. Here is another problem on number sequences that gives in to pattern analysis:

Exercise 0.1.3. Let $(u_0, u_1, u_2, \ldots)$ be a sequence of integers such that every $n \geq 1$ satisfies

$$u_n = nu_{n-1} + (-1)^n .$$

Prove that u_n is divisible by $u_{n-1} + u_{n-2}$ for each $n \geq 2$.

(After an idea of Euler, 1779)

Solution idea. In other words, we must prove that the fraction $\dfrac{u_n}{u_{n-1} + u_{n-2}}$ is an integer for each $n \geq 2$. Maybe we can explicitly compute this fraction?

First, let us set $s := u_0$. Then, the recursion lets us easily compute the first few entries of our sequence:

n	0	1	2	3	4	5	6
u_n	s	$s-1$	$2s-1$	$6s-4$	$24s-15$	$120s-76$	$720s-455$

.

Let us also divide them by $u_{n-1} + u_{n-2}$ and see what comes out:

n	0	1	2	3	4	5	6
$\dfrac{u_n}{u_{n-1} + u_{n-2}}$			1	2	3	4	5

.

Could it be any more obvious? We suspect that $\dfrac{u_n}{u_{n-1} + u_{n-2}} = n - 1$. In other words, we suspect that

$$u_n = (n-1)(u_{n-1} + u_{n-2}) \qquad \text{for each } n \geq 2. \tag{1}$$

It remains to prove this equality.

Indeed, fix $n \geq 2$. Since the equality we want to prove involves the three terms u_n, u_{n-1} and u_{n-2}, it appears reasonable to apply the recursive definition of our sequence once to n, obtaining

$$u_n = nu_{n-1} + (-1)^n , \tag{2}$$

and once again to $n - 1$, obtaining

$$u_{n-1} = (n-1)\, u_{n-2} + (-1)^{n-1} . \tag{3}$$

Wouldn't it be nice if we could obtain (1) from (2) and (3) by a bit of algebra?

Well, we can try. Ultimately, (2) and (3) are two linear equations in the three variables u_n, u_{n-1}, u_{n-2}, whereas (1) is another linear equation in the same variables. Linear algebra provides a surefire way (using Gaussian elimination) to check whether one linear equation follows from a set of others. We can also just try to cancel the $(-1)^n$ in (2) against the $(-1)^{n-1}$ in (3). Either way, we find the following:

Adding the equalities (2) and (3) together, we obtain

$$\begin{aligned} u_n + u_{n-1} &= \left(nu_{n-1} + (-1)^n\right) + \left((n-1)\,u_{n-2} + (-1)^{n-1}\right) \\ &= nu_{n-1} + (n-1)\,u_{n-2} + \underbrace{(-1)^n + (-1)^{n-1}}_{\substack{=0 \\ (\text{since } (-1)^n = -(-1)^{n-1})}} \\ &= nu_{n-1} + (n-1)\,u_{n-2}. \end{aligned}$$

Now, subtracting u_{n-1} from this equality, we obtain

$$u_n = nu_{n-1} + (n-1)\,u_{n-2} - u_{n-1} = (n-1)\,(u_{n-1} + u_{n-2}).$$

Thus, (1) is proven, and the claim of the problem follows. □

0.1.2. Giving names, algebraization

Mathematical problems are often stated in words. Even when these verbal descriptions are unambiguous and clear, it is often helpful to restate them in algebraic symbols, simply for the reason that such symbols can be algebraically manipulated (a craft you have been practicing since your high school days). I call this strategy *algebraization*. In school, it manifests itself as the "call the knowns $a, b, c, \ldots$ and the unknowns $x, y, z, \ldots$" method for solving word problems. In contest mathematics, it becomes a more sophisticated and creative undertaking, as you get to decide which objects to give names, what names to give them, how to order and organize them, etc.. Some choices end up more useful than others.

Algebraization is an art that cannot be fully learned, but let me give some examples.

Exercise 0.1.4. Let n be an even positive integer. Write the numbers $1, 2, \ldots, n^2$ in the squares of an $n \times n$ grid in such a way that the numbers in the k-th row, from left to right, are

$$(k-1)\,n + 1, \quad (k-1)\,n + 2, \quad \ldots, \quad (k-1)\,n + n.$$

Assume that the squares of the grid are colored in black and white, making sure that each row has exactly $\frac{n}{2}$ black squares and each column has exactly $\frac{n}{2}$ white squares. (For instance, the usual chessboard coloring satisfies this requirement, but so do some other colorings.)

Prove that the sum of the numbers in the white squares is equal to the sum of the numbers in the black squares.

(62nd Putnam contest 2001, Problem B1)

Solution idea. How can we algebraize the choice of black and white squares? The simplest way would be to give a name to the set of all squares (let's call it S), and to give a name to the set of all black squares (let's call it B), so that the set of all white squares becomes $S \setminus B$ (the set difference of S and B).

This can be useful, but there is a better way: First, we identify each square with a pair (i, j) of two integers $i, j \in \{1, 2, \ldots, n\}$ (namely, we let (i, j) stand for the square in the i-th row and the j-th column of our grid). Thus, the set of all squares is just the set of all such pairs. Next, for any $i, j \in \{1, 2, \ldots, n\}$, we define a number

$$a_{i,j} := \begin{cases} 1, & \text{if the square } (i,j) \text{ is black;} \\ 0, & \text{if the square } (i,j) \text{ is white.} \end{cases}$$

Thus, we have encoded our black-and-white coloring as a list of numbers $a_{i,j}$. (We could choose the numbers differently; this would require minor modifications in the solution that follows.)

Why is this encoding useful? For one, it lets us reformulate the assumptions of the problem as a list of equations. Namely, for each $i \in \{1, 2, \ldots, n\}$, the sum $\sum_{j=1}^{n} a_{i,j}$ equals the number of all black squares in the i-th row (since each black square (i, j) contributes an addend $a_{i,j} = 1$ to this sum, whereas each white square (i, j) contributes an addend $a_{i,j} = 0$ to it), and thus equals $\frac{n}{2}$ (since each row has exactly $\frac{n}{2}$ black squares). In other words, for each $i \in \{1, 2, \ldots, n\}$, we have[14]

$$\sum_{j=1}^{n} a_{i,j} = \frac{n}{2}. \tag{4}$$

Similarly, for each $j \in \{1, 2, \ldots, n\}$, we have

$$\sum_{i=1}^{n} a_{i,j} = \frac{n}{2}. \tag{5}$$

The integer written in the square (i, j) of our grid (as explained in the exercise) is

$$(i-1)\,n + j$$

(since it is the j-th entry of the i-th row). Thus, the sum of all numbers written in the grid is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left((i-1)\,n + j\right).$$

[14] All sums in this solution range from 1 to n. Thus, you can just write $\sum_j$ instead of $\sum_{j=1}^{n}$, provided that you explain this shorthand.

Furthermore, the sum of the numbers in all the black squares is

$$\sum_{i=1}^{n} \sum_{j=1}^{n} a_{i,j} \left((i-1) n + j \right)$$

(because the $a_{i,j}$ factor in this sum "turns off" all the white squares[15]). If we can compute these two sums, then we can also compute the sum of the numbers in all the white squares (as this is just the sum of all numbers minus the sum of the numbers in all the black squares), and compare it with the black sum; hopefully this will solve the exercise.

Let me compute the sum of the numbers in all the black squares, since this sum

[15] In fact, if (i,j) is a white square, then this factor $a_{i,j}$ is 0, and thus renders the entire addend $a_{i,j} \left((i-1) n + j \right)$ equal to 0. Meanwhile, if (i,j) is a black square, then this factor $a_{i,j}$ is 1, and thus the addend $a_{i,j} \left((i-1) n + j \right)$ is just $(i-1) n + j$, which is the number written in the square (i,j). Thus, the sum $\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{i,j} \left((i-1) n + j \right)$ includes all the numbers written in the black squares, and has a 0 for each white square.

is the harder one. It equals

$$\begin{aligned}
&\sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} a_{i,j}\left(\left(i-1\right)n+j\right)}_{\substack{=\sum_{j=1}^{n} a_{i,j}(i-1)n+\sum_{j=1}^{n} a_{i,j}j \\ \text{(by the standard rule } \sum_{j=1}^{n}\left(u_j+v_j\right)=\sum_{j=1}^{n} u_j+\sum_{j=1}^{n} v_j \\ \text{for finite sums)}}} \qquad \text{(as we saw above)} \\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} a_{i,j}\left(i-1\right)n+\sum_{j=1}^{n} a_{i,j}j\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\left(i-1\right)n+\sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}j \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{i,j}\left(i-1\right)n+\sum_{j=1}^{n}\sum_{i=1}^{n} a_{i,j}j \\
&\qquad \left(\begin{array}{c}\text{here, we interchanged the third and fourth} \\ \text{summation signs (i.e., we used the summation} \\ \text{rule } \sum_{i=1}^{n}\sum_{j=1}^{n} u_{i,j}=\sum_{j=1}^{n}\sum_{i=1}^{n} u_{i,j}\text{)}\end{array}\right) \\
&= \sum_{i=1}^{n}\underbrace{\left(\sum_{j=1}^{n} a_{i,j}\right)}_{\substack{=\frac{n}{2} \\ \text{(by (4))}}}\left(i-1\right)n+\sum_{j=1}^{n}\underbrace{\left(\sum_{i=1}^{n} a_{i,j}\right)}_{\substack{=\frac{n}{2} \\ \text{(by (5))}}}j \\
&= \sum_{i=1}^{n}\frac{n}{2}\left(i-1\right)n+\sum_{j=1}^{n}\frac{n}{2}j \\
&= \frac{1}{2}\left(\sum_{i=1}^{n} n\left(i-1\right)n+n\sum_{j=1}^{n} j\right). \qquad (6)
\end{aligned}$$

Meanwhile, the sum of all numbers written in the grid is

$$\sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} ((i-1)\,n + j)}_{= \sum_{j=1}^{n} (i-1)n + \sum_{j=1}^{n} j} \qquad \text{(as we saw above)}$$
$$= \sum_{i=1}^{n} \left(\sum_{j=1}^{n} (i-1)\,n + \sum_{j=1}^{n} j \right)$$
$$= \sum_{i=1}^{n} \underbrace{\sum_{j=1}^{n} (i-1)\,n}_{=n(i-1)n} + \underbrace{\sum_{i=1}^{n} \sum_{j=1}^{n} j}_{=n \sum_{j=1}^{n} j}$$
$$= \sum_{i=1}^{n} n\,(i-1)\,n + n \sum_{j=1}^{n} j.$$

This is obviously twice the right hand side of (6). In other words, the sum of all numbers written in the grid equals twice the sum of all numbers in the black squares. Hence, the sum of the numbers in the white squares (which is the former sum minus the latter sum) must equal the sum of all numbers in the black squares. The exercise is solved.

There are easier solutions, too, but the above one is very straightforward: Once you introduce the $a_{i,j}$, the argument "writes itself", as long as you remember how to manipulate sums and how to use the $a_{i,j}$ to restrict your sums to black squares. □

Remark 0.1.1. Exercise 0.1.4 can be generalized:

> Let $u_1, u_2, \ldots, u_n$ be n real numbers, and let $v_1, v_2, \ldots, v_n$ be n further real numbers. Write the numbers $u_i + v_j$ for all $i, j \in \{1, 2, \ldots, n\}$ in the squares of an $n \times n$-grid (putting the number $u_i + v_j$ into the square in the i-th row and the j-th column).
>
> Assume that the squares of the grid are colored in black and white, making sure that each row has exactly $\frac{n}{2}$ black squares and each column has exactly $\frac{n}{2}$ white squares.
>
> Prove that the sum of the numbers in the white squares is equal to the sum of the numbers in the black squares.

Exercise 0.1.4 is the particular case for $u_i = (i-1)\,n$ and $v_j = j$. The solution we gave above can be easily adapted to the generalization.

Exercise 0.1.5. Basketball star Shanille O'Keal keeps track of the number of successful free throws she has made in her first n attempts of the season. More specifically, for every positive integer n, we let $S(n)$ denote the number of successful throws among Shanille's first n throws, divided by n.

Early in the season, $S(n)$ was less than 80%, but by the end of the season, $S(n)$ was more than 80%. Was there necessarily a moment in between when $S(n)$ was exactly 80% ?

(65th Putnam contest 2004, Problem A1)

Solution idea. First, let us guess: Is the answer "yes" or "no"? If it is "yes", then we need to prove it. If it is "no", then a single counterexample (showing a sequence of hits and misses that leads to $S(n) > 80\%$ without ever encountering a $S(m) = 80\%$ moment along the way) is sufficient. The former would make a more interesting problem than the latter, so we suspect that the answer is "yes". (This kind of reasoning is not infallible, but helps us focus on the goal more likely to be right. Of course, you should not completely close your mind to the other possibility.)

Next, let us translate the problem into algebra. For any nonnegative integer n, we let $A(n)$ denote the number of successful free throws among Shanille's first n attempts. Then, $S(n) = \dfrac{A(n)}{n}$ for all positive integers n. Moreover, $A(n)$ is a nonnegative integer, and we have $A(0) = 0$ (since there are 0 successful attempts among the first 0 attempts). More importantly, $A(n)$ grows either by 0 or by 1 with each attempt (since a single attempt cannot result in more than one success). In other words, for each nonnegative integer n, we have

$$A(n+1) = A(n) \text{ or } A(n+1) = A(n) + 1.$$

Thus, in particular, for each nonnegative integer n, we have

$$A(n+1) \leq A(n) + 1. \tag{7}$$

Now, our condition says that $S(n)$ was $< 80\%$ early in the season but became $> 80\%$ at the end of the season. In other words, there are two positive integers n_1 and n_2 such that $n_1 < n_2$ and $S(n_1) < 80\%$ and $S(n_2) > 80\%$. Consider these n_1 and n_2. (Note that 80% means the number $\dfrac{80}{100}$; this is how mathematicians define percentages.)

Our goal (if our guess was correct) is to show that $S(n) = 80\%$ for some $n > 0$. To do so, we assume the contrary – i.e., we assume that $S(n)$ is never 80%. (Thus, we attempt a proof by contradiction.)

Consider the behavior of the number $S(n)$ as n gradually increases from n_1 by n_2 (growing by 1 at each step). At the beginning, $S(n)$ is $< 80\%$ (since $S(n_1) < 80\%$), but at the end, $S(n)$ is $\geq 80\%$ (since $S(n_2) > 80\%$). Hence, there must be a step in which $S(n)$ switches from being $< 80\%$ to being $\geq 80\%$ (since otherwise, $S(n)$ would stay $< 80\%$ forever). In other words, there exists an $m \in$

$\{n_1, n_1 + 1, n_1 + 2, \ldots, n_2 - 1\}$ such that

$$S(m) < 80\% \text{ but } S(m+1) \geq 80\%.$$

Consider this m. (Note that there may be several m's that qualify[16], but we just pick one and stick with it.)

Note that $S(m+1)$ cannot be 80% (since we assumed that $S(n)$ is never 80%), and thus from $S(m+1) \geq 80\%$ we obtain $S(m+1) > 80\%$.

Note what we did here: Instead of sticking with the two integers $n_1 < n_2$ satisfying

$$S(n_1) < 80\% \text{ and } S(n_2) > 80\%,$$

we have found two **consecutive** integers $m < m+1$ satisfying

$$S(m) < 80\% \text{ and } S(m+1) > 80\%.$$

This is a good step forward, since consecutive integers are more convenient to work with and, in particular, allow us to apply (7).

Now, let us expand the inequalities $S(m) < 80\%$ and $S(m+1) > 80\%$ and see what comes out.

Recall that $S(m) = \dfrac{A(m)}{m}$ (by the definition of $S(m)$) and thus $\dfrac{A(m)}{m} = S(m) < 80\% = \dfrac{80}{100} = \dfrac{4}{5}$. Cross-multiplying by the denominators in this inequality, we find

$$5 \cdot A(m) < 4m. \tag{8}$$

Similarly, from the inequality $S(m+1) > 80\%$, we obtain

$$5 \cdot A(m+1) > 4(m+1). \tag{9}$$

Furthermore, both sides of the inequality (8) are integers, and thus it entails

$$5 \cdot A(m) \leq 4m - 1 \tag{10}$$

(because if two integers a and b satisfy $a < b$, then $a \leq b - 1$).

However, (7) (applied to $n = m$) yields $A(m+1) \leq A(m) + 1$, and thus

$$5 \cdot \underbrace{A(m+1)}_{\leq A(m)+1} \leq 5 \cdot (A(m) + 1) = \underbrace{5 \cdot A(m)}_{\substack{\leq 4m-1 \\ \text{(by (10))}}} + 5$$
$$\leq 4m - 1 + 5 = 4m + 4 = 4(m+1).$$

This contradicts (9).

This contradiction shows that our assumption was false, and hence the problem is solved. □

[16] because $S(n)$ can increase, then decrease, then increase again

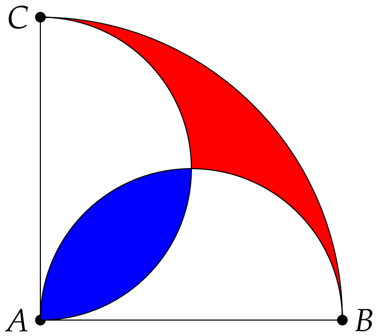
Here are a few more tricks you can learn from the above solution:

- If something in a problem can be described by an integer or by a non-integer, it is usually better to use the integer. Thus, in the above problem, the integer $A(n)$ was more useful than the non-integer $S(n)$, as it satisfied (7) and allowed us to use its integrality to get from (8) to (10).
- If two integers a and b satisfy $a < b$, then $a \leq b - 1$. Likewise, if two integers a and b satisfy $a > b$, then $a \geq b + 1$. Thus, strict inequalities between integers can be strengthened at the cost of their strictness. This is almost always useful when dealing with inequalities between integers. (But of course, it would not hold for rational or real numbers.)
- If a finite sequence of real numbers (in our above solution, these were the numbers $S(n)$ for $n_1 \leq n \leq n_2$) starts below a certain threshold (in our above solution, this was 80%) but ends above this threshold, then it must surpass this threshold at a certain step – i.e., it must contain two adjacent numbers of which the first is below the threshold and the second is above it. ("Above" should here be understood as allowing it to **equal** the threshold. Of course, in our above solution, we ruled this equality out by assuming that $S(n)$ is never 80%.)

0.1.3. The via negativa (subtraction)

The following two exercises come from completely different parts of mathematics, but their solutions exemplify the same idea.

Exercise 0.1.6. Consider the following picture:

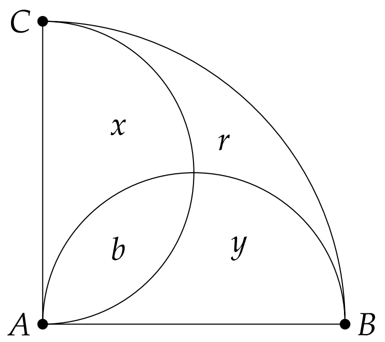


.

Compare the area of the red region with that of the blue region.

(Here, ABC is an isosceles right-angled triangle with right angle at A. We have erected semicircles with diameters AB and AC, both pointing into the inside of triangle ABC. We have also drawn a quarter-circle with center A and radius $AB = AC$ bordered by the points B and C. The red region is formed by removing the two semicircles from the quarter-circle. The blue region is the (set-theoretical) intersection of the two semicircles.)

Solution. The two half-circles subdivide the quarter-circle into four regions. We denote the areas of these four regions by b, r, x, y in the following way:



.

[17] Thus, the area of the red region is r, whereas the area of the blue region is b. We claim that $r = b$.

Indeed, let $a = |AB| = |AC|$. Thus, the quarter-circle has radius a; hence, its area is $\dfrac{1}{4}\pi a^2$. In other words,

$$b + r + x + y = \frac{1}{4}\pi a^2 \tag{11}$$

(since the quarter-circle has been subdivided into four regions with areas b, r, x, y, and thus its area is $b + r + x + y$).

However, the lower semicircle has diameter AB and thus radius $\dfrac{1}{2}\underbrace{|AB|}_{=a} = \dfrac{1}{2}a$. Hence, its area is $\dfrac{1}{2}\pi\left(\dfrac{1}{2}a\right)^2 = \dfrac{1}{8}\pi a^2$. In other words,

$$b + y = \frac{1}{8}\pi a^2 \tag{12}$$

(since the lower semicircle has been subdivided into two regions with areas b and y, and thus its area is $b + y$).

A similar argument (using the upper semicircle instead of the lower one) shows that $b + x = \dfrac{1}{8}\pi a^2$. Subtracting this equality from (11), we obtain $(b + r + x + y) -$

[17] Here is how to define b, r, x, y without referencing the picture: We shall refer to the semicircle with diameter AB as the *lower semicircle*. We shall refer to the semicircle with diameter AC as the *upper semicircle*. Let $\mathbb{A}(U)$ denote the area of any shape U. Now, we set

$$b = \mathbb{A}\left((\text{lower semicircle}) \cap (\text{upper semicircle})\right);$$
$$r = \mathbb{A}\left((\text{quarter-circle}) \setminus ((\text{lower semicircle}) \cup (\text{upper semicircle}))\right);$$
$$x = \mathbb{A}\left((\text{upper semicircle}) \setminus (\text{lower semicircle})\right);$$
$$y = \mathbb{A}\left((\text{lower semicircle}) \setminus (\text{upper semicircle})\right).$$

$(b+x) = \frac{1}{4}\pi a^2 - \frac{1}{8}\pi a^2$. This simplifies to $r+y = \frac{1}{8}\pi a^2$. Comparing this with (12), we obtain $r+y = b+y$. Hence, $r = b$. This solves the exercise. □

Exercise 0.1.7. Let $a_1, a_2, \ldots, a_n$ be n real numbers. Let $b_1, b_2, \ldots, b_n$ be the same n real numbers, arranged in a different order. Let s be a real number.

Let x be the number of all $i \in \{1, 2, \ldots, n\}$ such that $a_i \geq s > b_i$. Let y be the number of all $i \in \{1, 2, \ldots, n\}$ such that $b_i \geq s > a_i$. Prove that $x = y$.

Solution. An $i \in \{1, 2, \ldots, n\}$ satisfies $a_i \geq s > b_i$ if and only if it satisfies $a_i \geq s$ but not $b_i \geq s$ (because the inequality $s > b_i$ holds exactly when the inequality $b_i \geq s$ does not). But the definition of x shows that

$$\begin{aligned} x &= (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } a_i \geq s > b_i) \\ &= (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } a_i \geq s \text{ but not } b_i \geq s) \\ &\qquad \left(\begin{array}{c} \text{since } i \text{ satisfies } a_i \geq s > b_i \text{ if and only if} \\ \text{it satisfies } a_i \geq s \text{ but not } b_i \geq s \end{array}\right) \\ &= (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } a_i \geq s) \\ &\qquad - (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } a_i \geq s \text{ and } b_i \geq s) \end{aligned} \tag{13}$$

(because if we want to count some objects x that have some property $\mathcal{A}(x)$ but not some other property $\mathcal{B}(x)$, then we can first count all objects x that satisfy $\mathcal{A}(x)$, and then subtract the number of those x that satisfy both $\mathcal{A}(x)$ and $\mathcal{B}(x)$). An analogous computation shows that

$$\begin{aligned} y &= (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } b_i \geq s) \\ &\qquad - (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } b_i \geq s \text{ and } a_i \geq s). \end{aligned} \tag{14}$$

However, the n numbers $b_1, b_2, \ldots, b_n$ are just the n numbers $a_1, a_2, \ldots, a_n$ arranged in a different order. Thus, among the latter n numbers, there are equally many that are $\geq s$ as there are among the former n numbers. In other words,

$$\begin{aligned} &(\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } a_i \geq s) \\ &= (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } b_i \geq s). \end{aligned}$$

Moreover,

$$\begin{aligned} &(\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } a_i \geq s \text{ and } b_i \geq s) \\ &= (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } b_i \geq s \text{ and } a_i \geq s) \end{aligned}$$

(this is trivial: the logical connective "and" is commutative). Subtracting the latter equality from the former, we obtain

$$\begin{aligned} &= (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } a_i \geq s) \\ &\qquad - (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } a_i \geq s \text{ and } b_i \geq s) \\ &= (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } b_i \geq s) \\ &\qquad - (\text{the number of all } i \in \{1,2,\ldots,n\} \text{ such that } b_i \geq s \text{ and } a_i \geq s). \end{aligned}$$

In view of (13) and (14), we can rewrite this as $x = y$. Thus, Exercise 0.1.7 is solved. □

What do Exercise 0.1.6 and 0.1.7 have in common? Their solutions (at least the ones I showed above) both are instances of a general technique, which is *to view a thing as a difference of some larger (but simpler) things*. In Exercise 0.1.6, we rewrote the blue and red areas as differences of larger areas (e.g., we got to $r + y$ by subtracting $b + x$ from $b + r + x + y$, and we got r by subtracting y from $r + y$). In Exercise 0.1.7, we rewrote x and y as differences of larger (but simpler) numbers. Another instance of this technique is the classical solution of the quadratic equation[18]. I refer to this technique as the *via negativa*[19]. It is used quite frequently in combinatorics when counting things; the *principle of inclusion and exclusion* ([Grinbe20, §7.8]) takes it to the next level, but there are various elementary examples as well (e.g., [Grinbe20, Exercise 7.3.2]). I will not focus on enumerative combinatorics this year, so I will not elaborate on this.

(You can view the classical computational trick $14 \cdot 16 = 15^2 - 1$ as an instance of the via negativa as well.)

0.1.4. A harder problem

The final solved exercise for this worksheet is a more advanced example of "experiment and discover":

> **Exercise 0.1.8.** True or False: For any even positive integer n, it is possible to divide the set $\{1, 2, \ldots, n\}$ into pairs so that the sum of each pair is one less than a power of 2 (that is, can be written as $2^m - 1$ for some nonnegative integer m). (Each pair must consist of two distinct numbers, and each number in $\{1, 2, \ldots, n\}$ must belong to exactly one pair.)
>
> [**Example:** For $n = 6$, this can be done, and the pairs will be $\{1, 6\}$, $\{2, 5\}$ and $\{3, 4\}$.]
>
> ([WagVel20, Problem 23])

Solution idea. First of all, what answer should we expect? I'd bet on "True", because if the answer was "False", this would not be a good contest problem (the statement

[18] To recall: In order to solve the quadratic equation $x^2 + bx + c = 0$ (for fixed b and c), you rewrite $x^2 + bx + c$ as the difference $\left(x + \frac{b}{2}\right)^2 - \left(\frac{b^2}{4} - c\right)$. This is called "completing the square". Likewise, our solution of Exercise 0.1.6 can be viewed as "completing the half-circle", and in Exercise 0.1.7 we have "completed the count".

[19] This phrase originates in Christian theology, where it meant (very roughly) defining God by describing what God is not. It was recently repurposed by Nassim Taleb for any kind of decision or description through elimination of undesired features, something that has been done for millenia without an explicit name (Sherlock Holmes: "When you have eliminated the impossible, whatever remains, however improbable, must be the truth"). But a moniker can help bring it out into the foreground when it is used and remind of it when it can come useful.

would be uninteresting, for starters). This heuristic is not fully bulletproof (every once in a while, mischievous problem committees will propose a plausible-looking claim that is actually false for $n = 29$), but it helps us decide whether to start trying to prove or to disprove the claim (even if we might later redecide).

Having guessed that the answer is "True", we need to actually find these pairs. Let us do this for some small values of n:

n	the pairs
2	$\{1, 2\}$
4	$\{3, 4\}$, $\{1, 2\}$
6	$\{1, 6\}$, $\{2, 5\}$, $\{3, 4\}$
8	$\{7, 8\}$, $\{1, 6\}$, $\{2, 5\}$, $\{3, 4\}$
10	$\{5, 10\}$, $\{6, 9\}$, $\{7, 8\}$, $\{3, 4\}$, $\{1, 2\}$
12	$\{3, 12\}$, $\{4, 11\}$, $\{5, 10\}$, $\{6, 9\}$, $\{7, 8\}$, $\{1, 2\}$

Finding these pairings is actually quite easy: You start by identifying the partner of n (you can hope that it will be unique), then you identify the partner of $n-1$ (if it is not already "taken" by n), then the partner of $n-2$ (if it is not already "taken" by n or $n-1$), and so on, working your way downwards. This does not prove anything (what if some number finds no partner?), but at least it helps us fill the above table.

Time to find some patterns in the table. One pattern you might see from the table (it is even clearer from the method you use to construct the table) is that if the first pair[20] is $\{k, n\}$, then the next pairs will be $\{k+1, n-1\}$, $\{k+2, n-2\}$, ..., going all the way up to $\{k+\ell, n-\ell\}$ where ℓ is the largest integer such that $k+\ell < n-\ell$ (you cannot continue this pairing further, since that would repeat already-existing pairs). Explicitly, this ℓ will be $\dfrac{n-k-1}{2}$ (this is easy to see[21]). Thus, the last pair constructed so far will be $\left\{\dfrac{n+k-1}{2}, \dfrac{n+k+1}{2}\right\}$, and altogether we have paired up the numbers $k,\ k+1,\ k+2,\ \ldots,\ n$. It remains to pair up the remaining

[20]The use of the word "pair" for a 2-element set is a bit nonstandard (usually, "pair" means "ordered pair" in mathematics, and the two entries can be equal), but it is excused here since the problem statement sets a precedent.

[21]*Proof.* The inequality $k+\ell < n-\ell$ is equivalent to $\ell < \dfrac{n-k}{2}$ (this is just high-school algebra). Thus, the largest integer ℓ for which it holds is either $\dfrac{n-k-1}{2}$ or $\dfrac{n-k-2}{2}$, depending on whether $n-k$ is odd or even. It remains to show that $n-k$ is odd.

To show this, we recall that $\{k, n\}$ is a pair, so that we have $k+n = 2^m - 1$ for some $m > 0$ (why?). Thus, $k+n$ is odd (since 2^m is even), and therefore $n-k$ is odd as well (since $n-k = \underbrace{k+n}_{\text{odd}} - \underbrace{2k}_{\text{even}}$). This is what we needed to prove.

numbers $1, 2, \ldots, k-1$. Fortunately, $k-1$ is even[22]. Thus, to pair up $1, 2, \ldots, k-1$ is an instance of the same problem we are solving, but with the number $k-1$ instead of n (unless $k-1=0$, in which case we have nothing left to do). Thus, we have reduced our problem to a simpler variant thereof, in which the number n is replaced by the smaller number $k-1$. This yields a recursive algorithm for finding our pairing.

We still need to check one thing: We need to make sure that the partner of n can be found. In other words, we must prove that there exists some $k \in \{1, 2, \ldots, n-1\}$ such that $k+n$ is one less than a power of 2. What is this k ? Let us again make a table and try to identify a pattern:

n	2	4	6	8	10	12	14	16	18	20	22	24
k	1	3	1	7	5	3	1	15	13	11	9	7

.

Again, there is a visible pattern here: If n itself is a power of 2, then $k = n-1$. Then, as n increases, the respective k decreases by the same amount (e.g., as n grows from 16 to 18, the respective k falls from 15 to 13), until n hits the next power of 2. This can be restated explicitly: If 2^m is the largest power of 2 that satisfies $2^m \leq n$, then

$$k = 2^m - 1 - (n - 2^m). \tag{15}$$

Once this formula has been guessed, it is easy to verify that it works: If we define k by (15), then

$$k + n = 2^m - 1 - (n - 2^m) + n = \underbrace{2 \cdot 2^m}_{=2^{m+1}} - 1 = 2^{m+1} - 1$$

is clearly one less than a power of 2, so that $\{k, n\}$ is a valid pair. (You also have to check that this k belongs to $\{1, 2, \ldots, n-1\}$, but this is easy[23].)

Thus, we have solved the exercise, although we still need to write up our solution neatly. This should not be neglected: A good writeup of a proof is often **not** the same as a good explanation of the thought process by which the proof was found.

[22] *Proof.* We already found that $n-k$ is odd. Since n is even, this entails that k is odd. Hence, $k-1$ is even.

[23] *Proof.* Since 2^m is the largest power of 2 that satisfies $2^m \leq n$, we have $2^{m+1} > n$ (otherwise, 2^{m+1} would be an even larger power of 2 satisfying the same inequality). Dividing this inequality by 2, we obtain $2^m > n/2$. Since 2^m and $n/2$ are both integers (indeed, $n/2$ is an integer since n is even), we thus obtain $2^m \geq n/2 + 1$. Now,

$$k = 2^m - 1 - (n - 2^m) = 2 \cdot \underbrace{2^m}_{\geq n/2+1} - 1 - n \geq 2(n/2+1) - 1 - n = 1$$

and

$$k = 2^m - 1 - (n - 2^m) = 2 \cdot \underbrace{2^m}_{\leq n} - 1 - n \leq 2n - 1 - n = n - 1.$$

Combining these two inequalities, we find $k \in \{1, 2, \ldots, n-1\}$.

The latter can involve detours, leaps of faith and experiments, while the former should be streamlined, fully justified at each step (never leave the reader waiting for a justification!) and free of unnecessary information. When writing up a proof, it is unnecessary (and often inadvisable) to motivate each step heuristically, but it is important to make sure that each claim is fully justified using the previous claims.

Let us transform our above solution to Exercise 0.1.8 into such a clean writeup. This requires us to essentially read it backwards: First we define k by (15); then we prove that $\{k, n\}$ is a valid pair and so are $\{k+1, n-1\}$, $\{k+2, n-2\}$, ..., $\{k+\ell, n-\ell\}$ for $\ell = \frac{n-k-1}{2}$; finally we argue that the rest of the desired pairing can be obtained by performing the same construction for $k-1$ instead of n. And because this last step is recursive, we must organize our proof as a strong induction on n, so that the last step can be made by invoking the induction hypothesis. Here is the clean writeup that we obtain:

> *Solution to Exercise 0.1.8 (final copy).* We define a *valid pair* to mean a set $\{u, v\}$ consisting of two distinct integers such that $u + v$ is one less than a power of 2 (that is, such that $u + v = 2^m - 1$ for some nonnegative integer m). We define a *valid pairing* of the set $\{1, 2, \ldots, n\}$ to be a subdivision of this set $\{1, 2, \ldots, n\}$ into valid pairs. (A "*subdivision*" means a choice of valid pairs inside $\{1, 2, \ldots, n\}$ such that each element of $\{1, 2, \ldots, n\}$ belongs to exactly one of the chosen pairs.)[24]
>
> Thus, the problem is asking whether the set $\{1, 2, \ldots, n\}$ has a valid pairing for each even positive integer n.
>
> We claim that the answer to this question is "yes".[25]
>
> We will prove this claim by strong induction on n. We do not need a base case[26] (although the reader can easily check that it holds for $n = 2$).
>
> *Induction step:* Let n be an even positive integer. As our induction hypothesis, we assume that the set $\{1, 2, \ldots, n'\}$ has a valid pairing whenever $n' < n$ is an even positive integer. We must now show that the set $\{1, 2, \ldots, n\}$ has a valid pairing, too. We will construct such a pairing.
>
> We let 2^m be the largest power of 2 that satisfies $2^m \leq n$. (Why does this exist?[27]) Thus, $2^m \leq n$ but $2^{m+1} > n$ (since otherwise, 2^{m+1} would be an even larger power of 2 that would still satisfy $2^{m+1} \leq n$). Dividing the inequality $2^{m+1} > n$ by 2, we find $2^m > n/2$. Since 2^m and $n/2$ are

[24]We spent a whole paragraph just defining words. But it's worth it, since these words allow us to be clearer.

[25]Don't forget to state the answer when you are solving a "true or false?" problem.

[26]A strong induction needs no base case to work. See, e.g., [Grinbe23, Lecture 4, §1.9.4] for details.

[27]This is sufficiently easy that you don't need to justify it on the Putnam, but let me give a proof just in case: The powers of 2 are integers and are strictly increasing (i.e., we have $2^0 < 2^1 < 2^2 < \cdots$), which entails that they eventually surpass any given integer. Hence, they eventually surpass n, and from that point on remain larger than n. But they start out being $\leq n$ (since $2^0 = 1 \leq n$). Hence, there must be a last (i.e., largest) power of 2 that is $\leq n$.

integers (indeed, $n/2$ is an integer because n is even), we thus obtain $2^m \geq n/2+1$ (because if two integers a and b satisfy $a > b$, then $a \geq b+1$).

Now, set

$$k := 2^m - 1 - (n - 2^m) = \underbrace{2 \cdot 2^m}_{=2^{m+1}} - 1 - n = 2^{m+1} - 1 - n.$$

Thus, $k + n = 2^{m+1} - 1$ is one less than a power of 2. Furthermore, from

$$k = 2 \cdot \underbrace{2^m}_{\geq n/2+1} - 1 - n \geq 2 \cdot (n/2+1) - 1 - n = 1$$

and

$$k = 2 \cdot \underbrace{2^m}_{\leq n} - 1 - n \leq 2 \cdot n - 1 - n = n - 1,$$

we obtain $k \in \{1, 2, \ldots, n-1\}$, so that $k \in \{1, 2, \ldots, n\}$ and $k \neq n$. Hence, $\{k, n\}$ is a valid pair (since $k + n$ is one less than a power of 2).

Furthermore, $k + n = 2^{m+1} - 1$ is odd (since 2^{m+1} is even), so that $k = \underbrace{(k+n)}_{\text{odd}} - \underbrace{n}_{\text{even}}$ is odd and therefore $\underbrace{n}_{\text{even}} - \underbrace{k}_{\text{odd}}$ must also be odd. Thus, $n - k - 1$ is even. Hence, define the integer

$$\ell := \frac{n-k-1}{2}.$$

This is a nonnegative integer (nonnegative because $k \leq n-1$). For each $p \in \{0, 1, \ldots, \ell\}$, we have

$$(n-p) - (k+p) = n - k - 2 \underbrace{p}_{\leq \ell = \frac{n-k-1}{2}} \geq n - k - 2 \cdot \frac{n-k-1}{2} = 1 > 0,$$

so that $k + p < n - p$, and thus the set $\{k+p,\ n-p\}$ is a valid pair (since $(k+p) + (n-p) = k + n = 2^{m+1} - 1$ is one less than a power of 2). In other words, the $\ell + 1$ sets

$$\{k+0,\ n-0\},\ \{k+1,\ n-1\},\ \{k+2,\ n-2\},\ \ldots,\ \{k+\ell,\ n-\ell\}$$

are valid pairs. Altogether, these $\ell + 1$ sets contain (with no duplication) exactly the $2(\ell+1)$ integers from k to n (inclusive)[28].

If $k = 1$, then this shows that these $\ell + 1$ sets contain (with no duplication) exactly the $2(\ell+1)$ integers from 1 to n, and thus form a valid

[28]The easiest way to see this is as follows: The set $\{k, k+1, k+2, \ldots, n\}$ contains exactly $n - k + 1$

pairing of the set $\{1, 2, \ldots, n\}$ (since they are valid pairs). Hence, if $k = 1$, then the induction step is complete.

Now, consider the case when $k \neq 1$. In this case, $k - 1$ is a positive integer. Furthermore, this integer $k - 1$ is even (since k is odd) and smaller than n (since $k - 1 < k \leq n$). Hence, by the induction hypothesis, the set $\{1, 2, \ldots, k - 1\}$ has a valid pairing. This pairing covers the numbers $1, 2, \ldots, k - 1$. Adding to it the $\ell + 1$ valid pairs

$$\{k + 0,\ n - 0\}, \quad \{k + 1,\ n - 1\}, \quad \{k + 2,\ n - 2\}, \quad \ldots, \quad \{k + \ell,\ n - \ell\}$$

(which, as we know, cover the remaining numbers $k, k + 1, k + 2, \ldots, n$), we obtain a valid pairing of the set $\{1, 2, \ldots, n\}$. Thus, the induction step is complete. This completes the induction and thus the proof of our claim.

□

0.2. Class problems

The following problems are to be discussed during class.

Exercise 0.2.1. Let $(a_0, a_1, a_2, \ldots)$ be a sequence of real numbers defined recursively by $a_0 = 1$ and

$$a_n = n - a_{n-1} \qquad \text{for each } n \geq 1.$$

Compute a_{2023}.

Exercise 0.2.2. Let $\mathbb{N} = \{0, 1, 2, \ldots\}$. Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy

$$f(m + n) = f(m) + f(n) + mn \qquad \text{for all } m, n \in \mathbb{N}.$$

---

elements, i.e., exactly $2(\ell + 1)$ elements (since $n - k + 1 = 2 \cdot \underbrace{\frac{n - k - 1}{2}}_{= \ell} + 2 = 2\ell + 2 = 2(\ell + 1)$).

Hence, it can be subdivided into $\ell + 1$ two-element subsets by pairing up its two outermost elements (k and n) with each other, then pairing their neighbors ($k + 1$ and $n - 1$) with each other, then pairing their neighbors ($k + 2$ and $n - 2$) with each other, and so on. The $\ell + 1$ subsets obtained in this way are precisely

$$\{k + 0,\ n - 0\}, \quad \{k + 1,\ n - 1\}, \quad \{k + 2,\ n - 2\}, \quad \ldots, \quad \{k + \ell,\ n - \ell\}.$$

Hence, these $\ell + 1$ sets contain exactly the $2(\ell + 1)$ integers from k to n.

Exercise 0.2.3. Let $(a_0, a_1, a_2, \ldots)$ be a sequence of nonnegative real numbers that satisfies $a_0 = 1$ and

$$a_{n+1} = a_n + \sqrt{a_{n+1} + a_n} \qquad \text{for all } n \geq 0.$$

Prove that this sequence is uniquely determined by these requirements, and find an explicit formula for all its entries a_n.

(Bundeswettbewerb Mathematik (BWM) 2001, Round 1, Problem 2)

Exercise 0.2.4. Let A and B be two finite sets such that $|A| = |B|$. (The notation $|S|$ denotes the size of a set S, that is, the number of all elements of S.)

Prove that $|A \setminus B| = |B \setminus A|$.

Exercise 0.2.5. Let A be an $n \times n$-matrix with real entries. A *cross* of A shall mean the union of a column and a row of A. (Thus, any cross has $2n - 1$ entries.)

Let m be a real number. Assume that in each cross of A, the sum of all entries of the cross is $\geq m$. What is the smallest possible value that the sum of all n^2 entries of A can take?

(West German mathematics contest (BWM) 1971/72, Round 1, Problem 1)

Exercise 0.2.6. Solve the system of equations

$$\begin{cases} xy + x + y = 1; \\ yz + y + z = 2; \\ zx + z + x = 3 \end{cases}$$

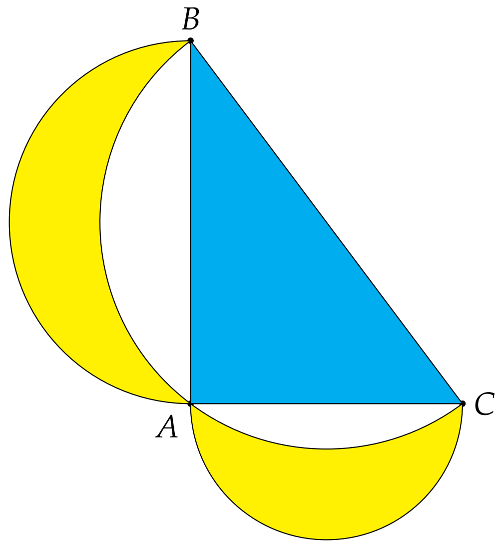
in real numbers x, y, z.

[**Hint:** Add something to $xy + x + y$ to get a polynomial that factors nicely.]

0.3. Homework exercises

Solve 5 of the 10 exercises below and upload your solutions on gradescope by October 3 before class.

Exercise 0.3.1. Consider the following picture:



Compare the area of the yellow region with that of the cyan region.

(Here, ABC is an arbitrary right-angled triangle with right angle at A. We have erected semicircles with diameters AB and AC, both pointing outside of triangle ABC, as well as a semicircle with diameter BC pointing inside it. The yellow region is formed by removing the latter semicircle from the union of the former two. The cyan region is just the interior of triangle ABC.)

(Hasan Ibn al-Haytham, ca. 1000)

Exercise 0.3.2. True or False: The pairing in Exercise 0.1.8 is unique (i.e., there is only one way to divide the set $\{1, 2, \ldots, n\}$ into pairs satisfying the conditions of Exercise 0.1.8).

Exercise 0.3.3.

(a) Would Exercise 0.1.5 still hold if we replaced "80%" by "50%"?

(b) What if we replaced "80%" by "60%"?

(c) What rational numbers r satisfying $0 < r < 1$ have the property that Exercise 0.1.5 still holds if "80%" is replaced by "r" throughout the exercise?

Exercise 0.3.4. Let $n > 3$ be an integer that is not prime. Show that we can find positive integers a, b, c such that $n = ab + bc + ca + 1$.

(49th Putnam contest 1988, Problem B1)

Exercise 0.3.5. Let k be a positive integer. Let b_k be the number $\underbrace{44\cdots4}_{k \text{ times}}\underbrace{88\cdots8}_{k-1 \text{ times}}9$ (written in base 10). For instance, $b_5 = 4444488889$.

Prove that b_k is a perfect square.

(A *perfect square* means the square of an integer.)

Exercise 0.3.6. Let k be a real number. Consider the sequence $(a_0, a_1, a_2, \ldots)$ of real numbers defined recursively by

$$a_0 = 1;$$
$$a_n = kn + (-1)^n a_{n-1} \qquad \text{for all } n \geq 1.$$

Compute a_{2023}.

(Simplified version of British Math Olympiad, Round 1, 1999/2000)

Exercise 0.3.7. Find a function $f : \mathbb{Z} \to \mathbb{Z}$ such that each $x \in \mathbb{Z}$ satisfies

$$f(f(x)) = x + 2 \qquad \text{but} \qquad f(x) \neq x + 1.$$

Exercise 0.3.8. Given three positive reals a, b, c, solve the system of equations

$$\begin{cases} x^2 = a + (y-z)^2; \\ y^2 = b + (z-x)^2; \\ z^2 = c + (x-y)^2 \end{cases}$$

in real numbers x, y, z.

Exercise 0.3.9. Let $(a_1, a_2, a_3, \ldots)$ be a sequence of reals such that all positive integers n satisfy

$$1a_1 + 2a_2 + \cdots + na_n = a_{n+1} - 1.$$

Set $c := a_1$. Find an explicit formula for a_n in terms of c.

Exercise 0.3.10. Let I be an interval on $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be a function. Assume that

$$f(a) + f(b) \geq 2f\left(\frac{a+b}{2}\right) \qquad \text{for all } a, b \in I.$$

(A function f satisfying this assumption is said to be *midpoint-convex* on I.)

(a) Prove that

$$f(a) + f(b) + f(c) + f(d) \geq 4f\left(\frac{a+b+c+d}{4}\right) \qquad \text{for all } a, b, c, d \in I.$$

(b) Prove that

$$f(a) + f(b) + f(c) \geq 3f\left(\frac{a+b+c}{3}\right) \qquad \text{for all } a, b, c \in I.$$

[**Hint:** To prove part **(b)**, apply part **(a)** with an appropriate choice of d.]

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