Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2023: Homework 9

Darij Grinberg

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1 Exercise 1

1.1 Problem

Consider the same game as in Exercise 9.1.2 on Worksheet 9, but with a minor difference: Now, each player is allowed to blow out 1, 2 or 4 candles (rather than $k \in \{1, 2, 3, 4, 5\}$ candles).

(a) Same question as before: Who wins (depending on n), assuming optimal play?

(b) What if, instead, each player can blow out 2 or 3 candles? Blowing out the second-to-last candle also counts as winning here (since it leaves the next player with no possible moves).

(c) Generalize part **(b)**: Fix an integer $m > 2$. Assume that each player can blow out an arbitrary number $k \in \{2, 3, \ldots, m-1\}$ of candles. Who wins?

1.2 Solution

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2 Exercise 2

2.1 Problem

Let A and B be two $n \times n$-matrices with complex entries such that $AB + A + B = 0$. Prove that $BA + B + A = 0$ as well.

2.2 Hint

One of the many facets of the infamous Invertible Matrix Theorem says the following: If X and Y are two $n \times n$-matrices that satisfy $XY = I_n$ (where I_n is the $n \times n$ identity matrix), then $YX = I_n$ as well.

2.3 Solution

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3 Exercise 3

3.1 Problem

Let n be a positive integer. Consider the 2^n many n-bitstrings (see Exercise 9.1.3 for their definition). Assume that 2^{n-1} of these 2^n bitstrings have been colored white (arbitrarily), while the remaining 2^{n-1} have been colored black. Prove that there exist at least 2^{n-1} pairs (w, b), where w is a white n-bitstring, where b is a black n-bitstring, and where w and b differ in exactly one position.

[**Example:** The two 4-bitstrings $(0, 1, 0, 0)$ and $(0, 1, 1, 0)$ differ in exactly one position (the third one).]

3.2 Solution

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4 Exercise 4

4.1 Problem

Let a, b, c be three positive reals satisfying $a^2 + b^2 + c^2 = 1$. Show that $\frac{bc}{a} + \frac{ca}{b} + \frac{ab}{c} \geq \sqrt{3}$.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let n be a positive integer such that $n \equiv -1 \mod 12$. Let s be the sum of all positive divisors of n. Prove that $12 \mid s$.

5.2 Solution

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6 Exercise 6

6.1 Problem

For each positive integer n, let $\operatorname{god}(n)$ denote the greatest odd divisor of n. For instance, $\operatorname{god}(24) = 3$ and $\operatorname{god}(25) = 25$ and $\operatorname{god}(30) = 15$.

Given two positive integers a and b. We define a sequence $(x_1, x_2, x_3, \ldots)$ of positive integers recursively by setting $x_1 = a$ and $x_2 = b$ and

$$x_n = \operatorname{god}(x_{n-1} + x_{n-2}) \qquad \text{for all } n \geq 3.$$

Prove that all entries of this sequence from a certain point on are equal to $\operatorname{god}(\gcd(a, b))$. In other words, prove that there exists a $k \geq 1$ such that $x_k = x_{k+1} = x_{k+2} = \cdots = \operatorname{god}(\gcd(a, b))$.

6.2 Solution

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7 Exercise 7

7.1 Problem

Let p and q be two coprime positive integers. Let $s = p + q$.

(a) Prove that $\{(ip) \% s \mid i \in \{1, 2, \ldots, s-1\}\} = \{1, 2, \ldots, s-1\}$. (That is, the $s-1$ remainders $(1p) \% s$, $(2p) \% s$, $\ldots$, $((s-1) p) \% s$ are just the $s-1$ numbers $1, 2, \ldots, s-1$ in some order.)

(b) Let $\mathbf{a} = (a_1, a_2, \ldots, a_{s-1})$ be an $(s-1)$-tuple of arbitrary objects (e.g., numbers). Given an integer $d \in \{1, 2, \ldots, s-1\}$, we say that $\mathbf{a}$ is *d-periodic* if each $i \in \{1, 2, \ldots, s-1-d\}$ satisfies $a_i = a_{i+d}$.

Assume that $\mathbf{a}$ is both p-periodic and q-periodic. Prove that $\mathbf{a}$ is 1-periodic (i.e., we have $a_1 = a_2 = \cdots = a_{s-1}$).

(c) Generalize part **(b)**: Let $\mathbf{a} = (a_1, a_2, \ldots, a_k)$ be a k-tuple of arbitrary objects, where $k \geq s-1$. Given an integer $d \in \{1, 2, \ldots, k\}$, we say that $\mathbf{a}$ is *d-periodic* if each $i \in \{1, 2, \ldots, k-d\}$ satisfies $a_i = a_{i+d}$.

Assume that $\mathbf{a}$ is both p-periodic and q-periodic. Prove that $\mathbf{a}$ is 1-periodic (i.e., we have $a_1 = a_2 = \cdots = a_k$).

7.2 Solution

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8 Exercise 8

8.1 Problem

Let $a_1, a_2, \ldots, a_n$ be n integers. A pair (i, j) of integers satisfying $1 \leq i < j \leq n$ will be called

- an *odd inversion* if a_i is odd, a_j is even, and $j - i$ is odd;
- an *even inversion* if a_i is odd, a_j is even, and $j - i$ is even.

Prove that the number of odd inversions is at least as large as the number of even inversions.

8.2 Solution

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9 Exercise 9

9.1 Problem

We write the numbers $1, 2, \ldots, n^2$ into the n^2 cells of an $n \times n$-chessboard (where $n > 1$). (Each number goes into exactly one cell.)

Prove that we can find two adjacent cells of this chessboard in which the two numbers differ by more than n. Here, "adjacent" means "have an edge or a vertex in common" (so a cell can have up to 8 other cells adjacent to it).

9.2 Solution

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10 Exercise 10

10.1 Problem

Continue with the notations of Exercise 9.1.3. Recall also the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$, defined by $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for each $n \geq 2$.

Prove that the total number of 1's summed over all lacunar n-bitstrings is

$$\frac{nf_{n+1} + 2(n+1) f_n}{5}.$$

10.2 Solution

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References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf