Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2026: Homework 8

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1 Exercise 1

1.1 Problem

Recall the Fibonacci sequence $(f_0, f_1, f_2, \ldots)$, defined by $f_0 = 0$ and $f_1 = 1$ and $f_n = f_{n-1} + f_{n-2}$ for all $n \geq 2$.

Let $n \in \mathbb{N}$. Let $P \in \mathbb{Q}[X]$ be a polynomial of degree $\leq n$ that satisfies $P(k) = f_k$ for each $k \in \{0, 1, \ldots, n\}$. Find $P(n+1)$.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let $P \in \mathbb{R}[X]$ be a polynomial such that $P(\cos\alpha) = P(\sin\alpha)$ for all $\alpha \in \mathbb{R}$. Prove that $P = Q(X^4 - X^2)$ for some $Q \in \mathbb{R}[X]$.

2.2 Solution

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3 Exercise 3

3.1 Problem

A polynomial $P \in \mathbb{Z}[X]$ is said to be *primitive* if the gcd of its coefficients is 1.

Let Q and R be two primitive polynomials in $\mathbb{Z}[X]$. Prove that their product QR is also primitive.

3.2 Hint

A bunch of integers are coprime if and only if they have no common prime divisor.

3.3 Solution

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4 Exercise 4

4.1 Problem

Let $n \in \mathbb{N}$. Prove that we have $X^2 + X + 1 \mid X^{2n} + X^n + 1$ in $\mathbb{Z}[X]$ if and only if $3 \nmid n$ in $\mathbb{Z}$.

4.2 Hint

First, define $N := X^2 + X + 1$, and show that $X^3 \equiv 1 \mod N$ in $\mathbb{Z}[X]$.

4.3 Solution

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Let $P \in \mathbb{Q}[X]$ be a polynomial of degree $\leq n$ that satisfies $P(k) = \dfrac{1}{k}$ for each $k \in \{1, 2, \ldots, n+1\}$. Show that $P(0) = \dfrac{1}{1} + \dfrac{1}{2} + \cdots + \dfrac{1}{n+1}$.

5.2 Hint

Take the derivative on both sides of the equality

$$XP - 1 = (X-1)(X-2)\cdots(X-(n+1)) \cdot Q$$

in the solution to Exercise 8.5.1 on Worksheet 8.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let n be an even positive integer. Let $u_1, u_2, \ldots, u_n$ be the n roots of the polynomial $X^n - nX + 1$. Prove that

$$\frac{1}{u_1+1} + \frac{1}{u_2+1} + \cdots + \frac{1}{u_n+1} = \frac{2n}{n+2}.$$

6.2 Hint

Find a degree-n polynomial with roots $\frac{1}{u_1+1}, \frac{1}{u_2+1}, \ldots, \frac{1}{u_n+1}$.

6.3 Solution

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7 Exercise 7

7.1 Problem

Let $P \in \mathbb{Z}[X]$ be a polynomial of even degree whose all coefficients (not counting the zero coefficients in front of powers that are larger than the degree) are odd. Prove that P has no rational root.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let $P = c_0X^0 + c_1X^1 + \cdots + c_nX^n$ be a polynomial in $\mathbb{Z}[X]$ (with $c_0, c_1, \ldots, c_n \in \mathbb{Z}$). Let $r \in \mathbb{Q}$ be a root of P. Prove that $c_nr^i + c_{n-1}r^{i-1} + \cdots + c_{n-i}r^0 \in \mathbb{Z}$ for each $i \in \{0, 1, \ldots, n\}$.

8.2 Solution

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9 Exercise 9

9.1 Problem

Let $a_0, a_1, \ldots, a_n$ be $n + 1$ pairwise distinct integers (where $n \in \mathbb{N}$). Prove that for any $s \in \mathbb{N}$, the number

$$\sum_{j=0}^{n} \frac{a_j^s}{\prod\limits_{k \neq j} (a_j - a_k)}$$

(where the "$\prod\limits_{k\neq j}$" sign means a product over all $k \in \{0, 1, \ldots, n\}$ satisfying $k \neq j$) is an integer.

9.2 Solution

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10 Exercise 10

10.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} i^{n+1} = \frac{n(n+1)!}{2}.$$

10.2 Hint

Alas, the polynomial $P := X^{n+1}$ has degree $n+1$, which is too high for an obvious application of Lagrange interpolation. Can you find a polynomial Q that has degree $\leq n$ but a sum $\sum\limits_{i=0}^{n} (-1)^{n-i} \binom{n}{i} Q(i)$ closely related to the sum on the left hand side?

10.3 Solution

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11 Exercise 11

11.1 Problem

Let $P \in \mathbb{R}[X]$ be a polynomial such that $P(x) \geq 0$ for all $x \in \mathbb{R}$. Prove that P can be written in the form $P = Q_1^2 + Q_2^2 + \cdots + Q_k^2$ for some polynomials $Q_1, Q_2, \ldots, Q_k \in \mathbb{R}[X]$.

11.2 Hint

First, show that if two polynomials P_1 and P_2 can be written in this form, then so can their product $P_1 P_2$. Next, show that every polynomial of the type $(X - z)(X - \overline{z})$ for two conjugate complex numbers $z = a + bi$ and $\overline{z} = a - bi$ can be rewritten in this form. Finally, use the Fundamental Theorem of Algebra on P.

11.3 Solution

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References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`