Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2023: Homework 7

Darij Grinberg

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1 Exercise 1

1.1 Problem

Let P be a polynomial (in one indeterminate) with integer coefficients. Let n be an odd positive integer. Let $a_1, a_2, \ldots, a_n$ be n integers that satisfy

$$P(a_i) = a_{i+1} \qquad \text{for each } i \in \{1, 2, \ldots, n\},$$

where we set $a_{n+1} := a_1$. Prove that $a_1 = a_2 = \cdots = a_n$.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let $(a_1, a_2, a_3, \ldots)$ be a sequence of reals such that $a_1 = 1$ and such that all positive integers k satisfy

$$1^2 a_1 + 2^2 a_2 + 3^2 a_3 + \cdots + k^2 a_k = \frac{k(k+1)}{2}(a_1 + a_2 + \cdots + a_k). \tag{1}$$

Find an explicit formula for a_n.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let n be a positive integer. Prove that either the number n or the number $3n$ (or both) begins with one of the digits 1, 2 and 9 when written in the decimal system.

(**Example:** If $n = 52$, then $3n = 156$ begins with a 1.)

3.2 Solution

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4 Exercise 4

4.1 Problem

For each positive integer n, we let $\operatorname{god}(n)$ denote the greatest odd divisor of n. For instance, $\operatorname{god}(24) = 3$ and $\operatorname{god}(25) = 25$ and $\operatorname{god}(30) = 15$.

Let n be a positive integer. Prove that

$$\operatorname{god}(n+1) + \operatorname{god}(n+2) + \cdots + \operatorname{god}(2n) = n^2.$$

4.2 Solution

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Let A be an $n \times n$-matrix whose all entries are odd integers.

Consider the product of all entries in each row and each column of A. Prove that the sum of these $2n$ products is congruent to $2n$ modulo 4.

5.2 Solution

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6 Exercise 6

6.1 Problem

Let n be a positive integer. Let A be an $n \times n$-matrix whose all entries belong to the set $\{0, 1\}$. Assume that A contains equally many 0's and 1's. Prove that A has two rows that contain the same number of 1's or two columns that contain the same number of 1's. (The "or" is not exclusive, so A can contain both.)

6.2 Solution

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7 Exercise 7

7.1 Problem

Consider a sequence $(a_0, a_1, a_2, \ldots)$ of non-zero reals that satisfies

$$a_n^2 = 1 + a_{n-1}a_{n+1} \qquad \text{for each } n \geq 1.$$

Prove that there exists a real number α such that $a_{n+1} = \alpha a_n - a_{n-1}$ for all $n \geq 1$.

7.2 Solution

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8 Exercise 8

8.1 Problem

Let n be a positive integer such that $4n + 1$ is a prime number. Prove that $4n + 1 \mid n^{2n} - 1$.

8.2 Solution

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9 Exercise 9

9.1 Problem

Let A be an $m \times n$-matrix whose entries are nonnegative reals and which has more columns than rows (that is, we have $n > m$). Assume that each column of A contains at least one positive entry.

Prove that A has at least one cell with a positive entry such that the sum of the entries in the row of this cell is larger than the sum of all entries in the column of this cell.

9.2 Solution

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10 Exercise 10

10.1 Problem

Prove that there exists no function $f : \mathbb{Z} \to \mathbb{Z}$ such that each $x \in \mathbb{Z}$ satisfies $f(f(x)) = x+1$.

[Note the contrast to Exercise 0.3.7 on worksheet #0.]

10.2 Solution

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References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`