Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2023: Homework 6

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1 Exercise 1

1.1 Problem

Recall the Euclidean algorithm discussed in Subsubsection 6.1.3. Let u and v be two positive integers. Prove that if we start the Euclidean algorithm in the state (u, v), then it will terminate after at most $\max\{u, v\}$ moves.

1.2 Remark

This is a sharp estimate, because if we start the Euclidean algorithm in the state $(n, 1)$, then it will take n moves to terminate:

$$(n,1) \mapsto (n-1,1) \mapsto (n-2,1) \mapsto \cdots \mapsto (1,1) \mapsto (1,0) \qquad (\text{or } \mapsto (0,1)).$$

On the other hand, the "grown-up" version of the Euclidean algorithm, which uses division with remainder instead of subtraction, terminates much faster: see, e.g., [Epasin83].

1.3 Solution

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2 Exercise 2

2.1 Problem

Consider the process whose states are the 4-tuples $(a, b, s, t) \in \{1, 2, 3, \ldots\}^4$ of positive integers, and whose moves are

$$(a, b, s, t) \mapsto (a, b - a, s, t + s) \qquad \text{if } b > a;$$
$$(a, b, s, t) \mapsto (a - b, b, s + t, t) \qquad \text{if } a > b.$$

Its end states are clearly the 4-tuples (a, b, s, t) with $a = b$.

Now, let u and v be two positive integers. We run this process starting with the state (u, v, u, v). Prove the following:

(a) This process will eventually terminate.

(b) The end state at which it terminates is a 4-tuple (g, g, s, t) with $g = \gcd(u, v)$ and $\dfrac{s+t}{2} = \operatorname{lcm}(u, v)$.

[**Example:** If $u = 6$ and $v = 9$, then the process runs as follows:

$$(6, 9, 6, 9) \mapsto (6, 3, 6, 15) \mapsto (3, 3, 21, 15),$$

and indeed we have $3 = \gcd(6, 9)$ and $\dfrac{21 + 15}{2} = 18 = \operatorname{lcm}(6, 9)$.]

2.2 Solution

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3 Exercise 3

3.1 Problem

Consider the following variant of Exercise 6.2.1: Chameleons can come in four colors, and change colors only when three chameleons of pairwise different colors meet (in which case they all take on the fourth color). (No more than three chameleons meet at a time.)

Encode each state as a 4-tuple (a, b, c, d), where a is the number of chameleons of the first color, and so on. Say that a state (a, b, c, d) is *winnable* if an appropriate sequence of meetings can transform it into a state where all chameleons have the same color.

(a) Prove that if a state (a, b, c, d) is winnable, then (at least) three of the integers a, b, c, d are congruent modulo 4.

(b) Is the converse true?

3.2 Solution

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4 Exercise 4

4.1 Problem

Consider the following variant of Exercise 6.2.1: We start with a red, b green and c blue chameleons. The rules of color change are the same as in Exercise 6.2.1, but the goal we want to achieve is not that all chameleons have the same color, but rather that the colors are equidistributed (i.e., there are equally many red, green and blue chameleons on the island).

Prove a simple necessary and sufficient criterion (in terms of a, b, c) for this goal to be achievable.

4.2 Solution

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5 Exercise 5

5.1 Problem

You have a bank account with 2023 dollars. You can split a bank account into two, but this operation costs you 1 dollar in fees. (Thus, you can split a bank account with n dollars into two bank accounts with p and q dollars each, where $n = p + q + 1$.) Can you use a sequence of such operations to end up with all of your accounts containing exactly 2 dollars each?

5.2 Solution

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6 Exercise 6

6.1 Problem

Consider Exercise 6.3.2 (the "pentagon game"), but now assume that the sum of all five integers is zero rather than positive. Assume furthermore that not all five integers are zero at the onset. Will the process go on forever, or will it terminate?

6.2 Solution

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7 Exercise 7

7.1 Problem

Four real numbers a, b, c, d are written on a whiteboard. Every hour, a ghost flies by and replaces them by the numbers $a - b$, $b - c$, $c - d$, $d - a$. Prove that eventually, at least one of the numbers will be too large (in absolute value) to fit on the whiteboard, unless the original four numbers a, b, c, d were all equal.

7.2 Solution

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8 Exercise 8

8.1 Problem

Consider a sequence $(a_0, a_1, a_2, \ldots)$ of non-zero reals that satisfies

$$a_n^2 = 1 + a_{n-1}a_{n+1} \qquad \text{for each } n \geq 1.$$

Prove that there exists a real number α such that $a_{n+1} = \alpha a_n - a_{n-1}$ for all $n \geq 1$.

(54th Putnam contest 1993, Problem A2)

8.2 Solution

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9 Exercise 9

9.1 Problem

Let m be a positive integer. Consider the ritual described in Exercise 6.4.1, but with 2^m points on the circle instead of 5 (and thus 2^m integers instead of 5).

Prove that after 2^m days, all the integers on the circle will be even.

9.2 Solution

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10 Exercise 10

10.1 Problem

Consider an n-tuple $(a_1, a_2, \ldots, a_n)$ of real numbers. We are allowed to make the following three kinds of moves:

- **S-moves:** We pick two adjacent entries a_i and a_{i+1} that are "out of order" (i.e., satisfy $a_i > a_{i+1}$), and swap them (i.e., replace them by a_{i+1} and a_i, respectively). This is called an *S-move*.
- **L-moves:** We pick two adjacent entries a_i and a_{i+1} that are "out of order" (i.e., satisfy $a_i > a_{i+1}$), and replace them by a_{i+1} and a_{i+1} (that is, we replace the larger entry by the smaller one). This is called an *L-move*.
- **U-moves:** We pick two adjacent entries a_i and a_{i+1} that are "out of order" (i.e., satisfy $a_i > a_{i+1}$), and replace them by a_i and a_i (that is, we replace the smaller entry by the larger one). This is called a *U-move*.

For instance, $(4, 3, 1, 2) \mapsto (3, 4, 1, 2)$ is an S-move, $(4, 3, 1, 2) \mapsto (3, 3, 1, 2)$ is an L-move, and $(4, 3, 1, 2) \mapsto (4, 4, 1, 2)$ is a U-move.

(a) Prove that if we keep applying S-moves and L-moves to our n-tuple (sequentially), then we eventually end up with a weakly increasing n-tuple (i.e., the process terminates).

(b) Is the same true if we allow all three kinds of moves?

10.2 Solution

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References

[Epasin83] P. W. Epasinghe, *Euclid's Algorithm and the Fibonacci Numbers*, Fibonacci Quarterly 23 (**1985**), issue 2, pp. 177–179.

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`