Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2023: Homework 5

Darij Grinberg

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1 Exercise 1

1.1 Problem

Let $a, b, c \in \mathbb{Z}$. Prove that:

(a) We have

$$\gcd(b, c) \cdot \gcd(c, a) \cdot \gcd(a, b) = \gcd(a, b, c) \cdot \gcd(bc, ca, ab).$$

(b) We have

$$\operatorname{lcm}(b, c) \cdot \operatorname{lcm}(c, a) \cdot \operatorname{lcm}(a, b) = \operatorname{lcm}(a, b, c) \cdot \operatorname{lcm}(bc, ca, ab).$$

(c) Assume that a, b, c are nonzero. Then,

$$\frac{\gcd(bc, ca, ab)}{\gcd(a, b, c)} = \frac{\operatorname{lcm}(bc, ca, ab)}{\operatorname{lcm}(a, b, c)} \in \mathbb{Z}.$$

1.2 Solution

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2 Exercise 2

2.1 Problem

Prove that $\gcd(a, b) = \gcd(a + b, \operatorname{lcm}(a, b))$ for any two integers a and b.

2.2 Solution

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3 Exercise 3

3.1 Problem

Fix a prime p. Let a and b be two integers satisfying $\gcd(a, b) = p$. What values can $\gcd(a^5, b^{13})$ take?

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $n, a, b, c \in \mathbb{N}$ such that n is odd. Prove that

$$\frac{(na)!\,(nb)!\,(nc)!}{a!b!c!\left((b+c)!\,(c+a)!\,(a+b)!\right)^{(n-1)/2}}$$

is an integer.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$(n+1) \operatorname{lcm}\left(\binom{n}{0}, \binom{n}{1}, \ldots, \binom{n}{n}\right) = \operatorname{lcm}(1, 2, \ldots, n+1).$$

5.2 Solution

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6 Exercise 6

6.1 Problem

Let n be any positive integer. Let a, b, c be three integers. Assume that $bc \mid a^n$ and $ca \mid b^n$ and $ab \mid c^n$. Show that $abc \mid (a+b+c)^{n+1}$.

6.2 Solution

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7 Exercise 7

7.1 Problem

Let k and n be any two positive integers. Prove that an expression of the form

$$\pm\frac{1}{k} \pm \frac{1}{k+1} \pm \frac{1}{k+2} \pm \cdots \pm \frac{1}{k+n}$$

(where each $\pm$ sign is either a $+$ or a $-$ sign) will never be an integer, no matter what the $\pm$ signs are.

7.2 Hint

Show that one of the numbers $k, k+1, \ldots, k+n$ has a higher 2-valuation than all of the others.

7.3 Solution

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8 Exercise 8

8.1 Problem

For any positive integer n, we let $\tau(n)$ denote the number of all positive divisors of n, and we let $\zeta(n)$ denote the product of all positive divisors of n. (For example, $\tau(12) = 6$ and $\zeta(12) = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 6 \cdot 12 = 1728$.)

(a) Prove that $\zeta(n) = n^{\tau(n)/2}$ for each integer $n \geq 2$.

(b) Let n and m be two positive integers satisfying $\zeta(n) = \zeta(m)$. Prove that $n = m$.

8.2 Solution

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9 Exercise 9

9.1 Problem

Let n be a positive integer. Let $d_1, d_2, \ldots, d_k$ be all positive divisors of n (including 1 and n itself, as usual). For each $i \in \{1, 2, \ldots, k\}$, let m_i be the number of all positive divisors of d_i. Prove that

$$m_1^3 + m_2^3 + \cdots + m_k^3 = (m_1 + m_2 + \cdots + m_k)^2 .$$

9.2 Hint

Feel free to use the classical identity

$$1^3 + 2^3 + \cdots + a^3 = (1 + 2 + \cdots + a)^2 ,$$

which holds for every $a \in \mathbb{N}$.

9.3 Solution

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10 Exercise 10

10.1 Problem

Let p be a prime. Let $n \in \mathbb{N}$. Set $n? := \prod_{k=1}^{n} k^k = 1^1 \cdot 2^2 \cdot \cdots \cdot n^n$. Prove that

$$v_p(n?) = \sum_{i \geq 1} p^i \cdot \frac{\lfloor n/p^i \rfloor \cdot (\lfloor n/p^i \rfloor + 1)}{2}.$$

10.2 Solution

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11 Exercise 11

11.1 Problem

Let $(u_1, u_2, u_3, \ldots)$ be a sequence of nonzero integers such that

$$\text{every } a, b \in \{1, 2, 3, \ldots\} \text{ satisfy } \gcd(u_a, u_b) = \left|u_{\gcd(a,b)}\right|.$$

(We have seen such a sequence in Exercise 2.9.6; another example is the Fibonacci sequence because of [Grinbe20, Exercise 3.7.2].)

Prove that there exists a sequence $|v_1, v_2, v_3, \ldots|$ of nonzero integers such that

$$\text{each } n \in \{1, 2, 3, \ldots\} \text{ satisfies } u_n = \prod_{d|n} v_d.$$

Here, the symbol "$\prod\limits_{d|n}$" means a product over all positive divisors d of n. Thus, the equality $u_n = \prod\limits_{d|n} v_d$. means

$$\begin{aligned}
u_1 &= v_1 && \text{for } n = 1; \\
u_2 &= v_1 v_2 && \text{for } n = 2; \\
u_3 &= v_1 v_3 && \text{for } n = 3; \\
u_4 &= v_1 v_2 v_4 && \text{for } n = 4; \\
u_5 &= v_1 v_5 && \text{for } n = 5; \\
u_6 &= v_1 v_2 v_3 v_6 && \text{for } n = 6;
\end{aligned}$$

and so on.

11.2 Solution

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References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`