Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2023: Homework 4

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1 Exercise 1

1.1 Problem

Let n be a positive integer. Let u be the number of pairs (j, k) of positive integers satisfying $\frac{1}{j} + \frac{1}{k} = \frac{1}{n}$.

Prove that u is the number of all positive divisors of n^2.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let $n \geq 2$ be an integer. Simplify the product $\prod_{k=2}^{n} \frac{k^3 - 1}{k^3 + 1}$.

2.2 Solution

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. Find a closed-form expression for the sum $\sum\limits_{k=1}^{n} (k^2 + 1) \cdot k!$.

3.2 Solution

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4 Exercise 4

4.1 Problem

Prove that

$$a(a-b)(a-c) + b(b-c)(b-a) + c(c-a)(c-b) \geq 0$$

for any three nonnegative reals a, b, c.

4.2 Hint

Don't look for a factorization! This polynomial does not have a nontrivial factorization.

4.3 Solution

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5 Exercise 5

5.1 Problem

Prove that

$$\left|\frac{a}{b-c}\right| + \left|\frac{b}{c-a}\right| + \left|\frac{c}{a-b}\right| \geq 2$$

for any three distinct reals a, b, c.

5.2 Hint

For symmetry reasons, you can WLOG assume that the absolute value $|c|$ is the smallest of $|a|$, $|b|$, $|c|$.

5.3 Solution

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6 Exercise 6

6.1 Problem

Let a, b, c be three real numbers such that $(a+b+c)^3 = a^3+b^3+c^3$. Prove that $(a+b+c)^n = a^n+b^n+c^n$ for each odd positive integer n.

6.2 Solution

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7 Exercise 7

7.1 Problem

Let n be an even positive integer. Find (and prove) the expanded form of

$$\underbrace{\left(1-x+x^2-x^3\pm\cdots+x^n\right)}_{=\sum\limits_{i=0}^{n}(-1)^i x^i} \cdot \underbrace{\left(1+x+x^2+\cdots+x^n\right)}_{=\sum\limits_{i=0}^{n} x^i}.$$

7.2 Solution

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8 Exercise 8

8.1 Problem

Let $n > 1$ be an integer. Factor the polynomial

$$\left(1+x+x^2+\cdots+x^n\right)^2 - x^n$$

as a product of two non-constant polynomials.

8.2 Solution

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9 Exercise 9

9.1 Problem

Let $f(x)$ be a polynomial with integer coefficients. Consider the plot of this polynomial (in the usual xy-plane). Let P_1 and P_2 be two distinct points on this plot. Assume that the coordinates of P_1 and P_2 as well as the distance $|P_1P_2|$ are integers. Prove that the line P_1P_2 is parallel to the x-axis.

9.2 Hint

Use Exercise 4.1.2 on worksheet #4. You can use the fact ([Grinbe20, Exercise 9.3.2 for $k = 2$]) that if an integer n is the square of a rational number, then n is the square of an integer.

9.3 Solution

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10 Exercise 10

10.1 Problem

Let n be a positive integer. Let $g(x)$ denote the polynomial $-(x^1 + x^2 + \cdots + x^n)$. Let $h(x)$ denote the polynomial

$$\sum_{k=0}^{n} (g(x))^k = (g(x))^0 + (g(x))^1 + (g(x))^2 + \cdots + (g(x))^n .$$

Prove that the coefficients of the powers $x^2, x^3, \ldots, x^n$ in $h(x)$ are 0.

10.2 Solution

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References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`