Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2023: Homework 3

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1 Exercise 1

1.1 Problem

Let a, b, c be three odd integers. Prove that at least one of the three integers $ab-1$, $bc-1$ and $ca-1$ is divisible by 4.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let S be a 10-element subset of the set $\{1, 2, \ldots, 100\}$. Prove that there exist two disjoint nonempty subsets A and B of S such that $\sum\limits_{a \in A} a = \sum\limits_{b \in B} b$. (Note that A and B are allowed to have any positive sizes, including 1.)

[**Example:** If $S = \{3, 9, 13, 19, 26, 60, 74, 80, 84, 94\}$, then $\sum\limits_{a \in A} a = \sum\limits_{b \in B} b$ is satisfied for $A = \{3, 9, 94\}$ and $B = \{26, 80\}$ (as well as for various other choices).]

2.2 Solution

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3 Exercise 3

3.1 Problem

A number of people have been settled in n apartments $B_1, B_2, \ldots, B_n$ (with each person settled in exactly one apartment). (Roommates are allowed.) Now, all these people are removed from their apartments and resettled in $n+1$ new apartments $C_1, C_2, \ldots, C_{n+1}$ in such a way that none of these $n+1$ new apartments stays empty.

A person is said to have *gained space* if he has fewer roommates after the resettlement than he used to have before.

Prove that at least two people have gained space.

3.2 Solution

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4 Exercise 4

4.1 Problem

Consider any six points on the circumference of a circle with radius 1. Prove that some two of these six points have distance ≤ 1.

4.2 Solution

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5 Exercise 5

5.1 Problem

Each lattice point in the Euclidean plane has been colored with one of n colors. Prove that there exists a rectangle whose four vertices are lattice points all having the same color.

5.2 Solution

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6 Exercise 6

6.1 Problem

Let $(a_0, a_1, a_2, \ldots)$ be the sequence of integers defined recursively by $a_0 = 0$ and $a_1 = a_2 = 1$ and

$$a_n = a_{n-1}a_{n-2} + a_{n-3} \qquad \text{for all } n \geq 3.$$

Let m be a positive integer. For each $i \in \mathbb{N}$, let b_i be the remainder of a_i upon division by m. Prove that the sequence $(b_0, b_1, b_2, \ldots)$ is purely periodic.

6.2 Solution

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7 Exercise 7

7.1 Problem

Let $n \in \mathbb{N}$. Consider a $2n \times 2n$-table. Assume that at most $3n$ of its $(2n)^2$ many squares have been colored black. Prove that it is possible to remove n rows and n columns from the table in such a way that no black squares remain in the table.

7.2 Hint

When $n \geq 2$, find two rows and two columns that contain at least 6 black squares among them.

7.3 Solution

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8 Exercise 8

8.1 Problem

Let n be a positive integer.

(a) Let S be an $(n+1)$-element subset of $[2n-1]$. Prove that we can find three distinct elements u, v, w of S such that $u+v=w$.

(b) Is this still true if we let S be a subset of $[2n]$ instead of a subset of $[2n-1]$?

8.2 Solution

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9 Exercise 9

9.1 Problem

Let $n \geq 2$ be an integer. Let A be an $n \times n$-matrix with real entries.

Assume that in each row of A, the sum of the two largest entries is positive.

Is it possible that in each column of A, the sum of the two largest entries is negative?

9.2 Solution

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10 Exercise 10

10.1 Problem

Let A be a rectangular matrix with real entries. Assume that the entries in each **row** of A are weakly increasing from left to right. Now we sort the entries in each **column** of A so that they become weakly increasing from top to bottom. (Each entry stays within its column.) Thus, we obtain a new matrix B. Prove that the entries in each **row** of B are still weakly increasing from left to right.

[**Example:** If $A = \begin{pmatrix} 1 & 3 & 6 \\ 2 & 2 & 5 \\ 1 & 2 & 3 \end{pmatrix}$, then $B = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 5 \\ 2 & 3 & 6 \end{pmatrix}$.]

10.2 Solution

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References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021.
`https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`