Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2023: Homework 2

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1 Exercise 1

1.1 Problem

Let b be an odd positive integer. Prove that the second-to-last digit of b^2 (in decimal notation) is even.

[**Example:** The second-to-last digit of $37^2 = 1369$ is 6, which is even.]

1.2 Solution

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2 Exercise 2

2.1 Problem

Let p be a prime. Let $k \in \mathbb{N}$.

(a) Prove that

$$\binom{k}{p-1} \equiv \begin{cases} 1, & \text{if } k \equiv -1 \mod p; \\ 0, & \text{if } k \not\equiv -1 \mod p \end{cases} \mod p.$$

(b) Prove that
$$\binom{k}{p} \equiv \left\lfloor \frac{k}{p} \right\rfloor \mod p.$$

2.2 Solution

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3 Exercise 3

3.1 Problem

The number 0 is written on a blackboard. You are allowed to take a number x written on the blackboard and replace it by either $3x+1$ or $\lfloor x/2 \rfloor$ (you choose which). Prove that you can obtain any nonnegative integer by a sequence of such operations.

[**Example:** If you want to obtain 19, you can proceed as follows: $0 \overset{A}{\to} 1 \overset{A}{\to} 4 \overset{A}{\to} 13 \overset{B}{\to} 6 \overset{A}{\to} 19$. Here, an "$\overset{A}{\to}$" arrow means a step of the form $x \mapsto 3x+1$, whereas a "$\overset{B}{\to}$" arrow means a step of the form $x \mapsto \lfloor x/2 \rfloor$.]

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $n > 3$ be an integer. Consider a collection of n stones, weighing $1, 2, \ldots, n$ pounds, respectively. For what values of n is it possible to subdivide this collection into three piles that have the same total masses? (The piles need not have the same number of stones, only the same total weight.)

[**Example:** This is possible for $n = 5$ (since $1 + 4 = 2 + 3 = 5$) and for $n = 8$ (since $1 + 3 + 8 = 2 + 4 + 6 = 5 + 7$) but not for $n = 7$ (why not?).]

4.2 Solution

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5 Exercise 5

5.1 Problem

Let p and q be two coprime positive integers. Prove that

$$\sum_{k=0}^{p-1} \left\lfloor \frac{kq}{p} \right\rfloor = \frac{(p-1)(q-1)}{2}.$$

5.2 Solution

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6 Exercise 6

6.1 Problem

Let $P(x)$ and $(s_0, s_1, s_2, \ldots)$ be as in Exercise 2.6.1 on worksheet #2. Prove that

$$\gcd(s_a, s_b) = \left|s_{\gcd(a,b)}\right| \qquad \text{for any } a, b \in \mathbb{N}.$$

6.2 Hint

Show that $\gcd(s_a, a_b) = \gcd(s_a, s_{b-a})$ whenever $a \leq b$; then argue by strong induction as in the proof of the Bezout theorem ([Grinbe20, proof of Theorem 3.4.5]).]

6.3 Solution

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7 Exercise 7

7.1 Problem

For any $n \in \mathbb{N}$, set

$$u_n := \sum_{k=0}^{n} \frac{n!}{k!} = \frac{n!}{0!} + \frac{n!}{1!} + \cdots + \frac{n!}{n!}.$$

(This is an integer, since $\dfrac{n!}{k!} = (k+1)(k+2)\cdots n$ for any $k \leq n$. Combinatorially, u_n is the number of all tuples of distinct elements from the set $\{1, 2, \ldots, n\}$. For instance, for $n = 2$, we have $u_2 = 5$, counting the tuples $()$, (1), (2), $(1, 2)$ and $(2, 1)$.)

Prove that if x and y are two nonnegative integers, then $u_x - u_y$ is divisible by $x - y$.

7.2 Hint

First, express u_n recursively through u_{n-1}.

7.3 Solution

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8 Exercise 8

8.1 Problem

Let p be a prime such that $p \equiv 3 \mod 4$.

(a) Prove that there exists no $c \in \mathbb{Z}$ such that $c^2 \equiv -1 \mod p$.

(b) Prove that if $a, b \in \mathbb{Z}$ are two integers satisfying $a^2 + b^2 \equiv 0 \mod p$, then a and b are multiples of p.

(c) Prove that there exist no two rational numbers a and b such that $a^2 + b^2 = p$.

8.2 Hint

For part **(a)**, apply Exercise 2.8.4 **(d)** on worksheet #2 and note that $(p-1)/2$ is odd. For part **(b)**, use modular inverses modulo p. Note that part **(c)** generalizes Exercise 2.8.1.

8.3 Solution

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9 Exercise 9

9.1 Problem

Prove that there exist infinitely many primes that are congruent to 1 modulo 4.

9.2 Hint

You can use Exercise 8 **(a)**.

9.3 Solution

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10 Exercise 10

10.1 Problem

Let n be a positive integer. Prove that there exist integers a and b such that $n \mid 4a^2+9b^2-1$.

10.2 Hint

Write n as $n = 2^i m$, where m is odd. Now find a solution to the congruence $4a^2 + 9b^2 \equiv 1 \mod 2^i$ and a solution of the congruence $4a^2 + 9b^2 \equiv 1 \mod m$. Combine them using the Chinese Remainder Theorem to obtain a solution of $4a^2 + 9b^2 \equiv 1 \mod n$.

10.3 Solution

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References

[Grinbe20] Darij Grinberg, *Math 235: Mathematical Problem Solving*, 10 August 2021. `https://www.cip.ifi.lmu.de/~grinberg/t/20f/mps.pdf`