Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2023: Homework 1

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October 18, 2023

1 Exercise 1

1.1 Problem

The *Lucas sequence* is the sequence $(\ell_0, \ell_1, \ell_2, \ldots)$ of integers which is defined recursively by

$$\ell_0 := 2, \qquad \ell_1 := 1, \qquad \text{and} \qquad \ell_n := \ell_{n-1} + \ell_{n-2} \text{ for all } n \geq 2.$$

(Thus, this sequence satisfies the same recursive equation as the Fibonacci sequence, differing only in the starting value $\ell_0 \neq f_0$. For instance, $\ell_2 = 3$ and $\ell_4 = 4$ and $\ell_5 = 7$.)

(a) Prove that $\ell_n = f_{n-1} + f_{n+1}$ for each $n \geq 1$.

(b) Prove that $\ell_n^2 - 5f_n^2 = 4 \cdot (-1)^n$ for each $n \geq 0$.

1.2 Solution

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2 Exercise 2

2.1 Problem

Let n be a positive integer. For each $k \in \{1, 2, \ldots, n-1\}$, we let

$$a_k := (n-k) \prod_{i=0}^{k-2} (n-i) = (n-k)(n-k+2)(n-k+3) \cdots n.$$

Prove that

$$\sum_{k=1}^{n-1} a_k = n! - 1.$$

2.2 Hint

For convenience, rename a_k as $a_{n,k}$ to stress the dependence on n.

2.3 Solution

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3 Exercise 3

3.1 Problem

Let $n \in \mathbb{N}$. Prove that

$$\sum_{r=1}^{n} \frac{1}{r} \binom{n}{r} = \sum_{r=1}^{n} \frac{2^r - 1}{r}.$$

3.2 Solution

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4 Exercise 4

4.1 Problem

Let n and p be positive integers such that $p \leq 2n$. Prove that

$$\sum_{k=p}^{n} 2^k k \binom{2n-k-1}{n-1} = 2^p n \binom{2n-p}{n}.$$

4.2 Hint

If you induct on p, the base case will be hard. Try to induct on $n-p$ instead! (If $p>n$, then both sides are 0, so you can assume $n-p$ to be nonnegative.)

4.3 Solution

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5 Exercise 5

5.1 Problem

Let S be a set of nonnegative rational numbers. Assume the following.

1. We have $0 \in S$.
2. For any $x \in S$, we have $\dfrac{1}{x+1} \in S$ and $\dfrac{x}{x+1} \in S$.

Prove that the set S contains all rational numbers in the interval $[0,1]$.

5.2 Solution

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6 Exercise 6

6.1 Problem

We will say that a finite sequence $(x_1, x_2, \ldots, x_n)$ of integers is *rude* if it there exist no three integers i, j, k with

$$1 \leq i < j < k \leq n \qquad \text{and} \qquad (x_i + x_j = 2x_k \text{ or } x_j + x_k = 2x_i).$$

(In other words, a finite sequence is rude if and only if no two of its entries have their average written anywhere to the left of them both or to the right of them both.)

Prove that for every integer $n \geq 2$, there exist exactly two rude permutations of the sequence $(1, 2, \ldots, n)$, namely $(1, 2, \ldots, n)$ itself and its reversal $(n, n-1, \ldots, 1)$.

6.2 Solution

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7 Exercise 7

7.1 Problem

Let n be a positive integer. In a tournament, 2^{n-1} contestants participate, with each pair of (distinct) contestants playing exactly one round against each other ("round-robin tournament"). Each round is won by exactly one player (there are no ties).

Prove that we can find n distinct contestants $c_1, c_2, \ldots, c_n$ such that for each $i < j$, the contestant c_i wins against c_j.

7.2 Solution

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8 Exercise 8

8.1 Problem

A country has n towns (with $n \geq 1$), arranged along a linear road running from left to right. Each town has a *left bulldozer* (standing on the road to the left of the town and facing left) and a *right bulldozer* (standing on the road to the right of the town and facing right). The sizes of the $2n$ bulldozers are distinct. Every time when a left and right bulldozer confront each other, the larger bulldozer pushes the smaller one off the road. On the other hand, bulldozers are unprotected at their rears; so, if a bulldozer reaches the rear-end of another one, the first one pushes the second one off the road, regardless of their sizes.

For any two towns P and Q, we say that town P *dominates* town Q if the bulldozer of P that is facing in the direction of Q can move over to Q without getting pushed off the road.

Prove that there is exactly one town that is not dominated by any other town.

[**Example:** Here is one possibility for $n = 5$:

$$\underline{\quad 7A3 \qquad 5B4 \qquad 1C9 \qquad 2D6 \qquad 8E10 \quad} ,$$

where A, B, C, D, E are the five towns and where each number stands for the size of the corresponding bulldozer. It is easy to check that in this configuration, town A dominates no other town; town B dominates towns A, C and D; town C dominates D and E; town D dominates no other towns; town E dominates town D. Thus, the unique undominated town is B.]

8.2 Hint

What happens if we remove the town with the largest bulldozer?

8.3 Solution

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9 Exercise 9

9.1 Problem

Let k and n be two nonnegative integers. Let S be a set with size $|S| \geq k(n+1) - 1$. Assume that each n-element subset of S is colored either red or green. Prove that there exist k pairwise disjoint n-element subsets of S that have the same color.

9.2 Solution

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10 Exercise 10

10.1 Problem

Let n be a positive integer. You have n boxes, and each box contains a non-negative number of pebbles. In each move, you are allowed to take two pebbles from an arbitrary box, throw away one of the pebbles and put the other pebble in another box. (You can freely choose these two boxes.) An initial configuration of pebbles is called *solvable* if it is possible to reach a configuration with no empty box, in a finite (possibly zero) number of moves.

Prove that a configuration $(a_1, a_2, \ldots, a_n)$ (that is, a configuration in which box 1 has a_1 pebbles, box 2 has a_2 pebbles, and so on) is solvable if and only if

$$\left\lceil \frac{a_1}{2} \right\rceil + \left\lceil \frac{a_2}{2} \right\rceil + \cdots + \left\lceil \frac{a_n}{2} \right\rceil \geq n.$$

Here, $\lceil x \rceil$ denotes the *ceiling* of a real number x (that is, the smallest integer that is $\geq x$).

[**Example:** The configuration $(3, 0, 6, 0, 0)$ is solvable. Indeed, we can take two pebbles from the first box and move one of them to the second, obtaining $(1, 1, 6, 0, 0)$; then we can take two pebbles from the third box and move one of them to the fourth, obtaining $(1, 1, 4, 1, 0)$; and finally take two pebbles from the third box and move one of them to the fifth, obtaining $(1, 1, 2, 1, 1)$.]

10.2 Hint

For a configuration $(a_1, a_2, \ldots, a_n)$, define its *size* to be $a_1 + a_2 + \cdots + a_n$ (that is, the total number of pebbles), and define its *level* to be $\left\lceil \frac{a_1}{2} \right\rceil + \left\lceil \frac{a_2}{2} \right\rceil + \cdots + \left\lceil \frac{a_n}{2} \right\rceil$. Note that the size decreases by 1 with each move, whereas the level either stays the same or decreases by 1 (why?). This suggests making moves that don't decrease the level unless absolutely necessary. When is it necessary?

10.3 Solution

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References