Drexel University, Department of Mathematics

Math 235: Mathematical Problem Solving, Fall 2023: Homework 0

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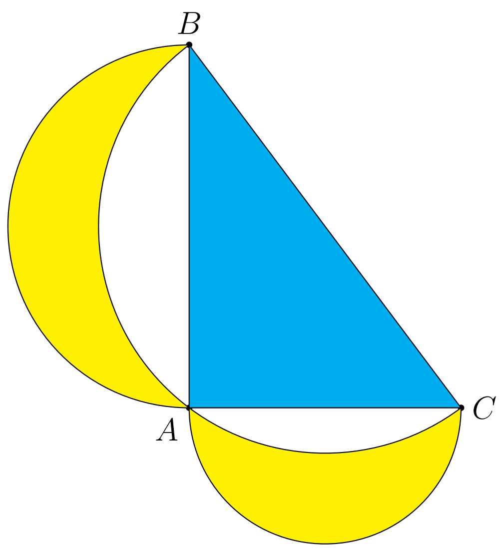
Please solve 5 of the 10 exercises!
Deadline: October 3, 2023

1 Exercise 1

1.1 Problem

Consider the following picture:



Compare the area of the yellow region with that of the cyan region.

(Here, ABC is an arbitrary right-angled triangle with right angle at A. We have erected semicircles with diameters AB and AC, both pointing outside of triangle ABC, as well as a semicircle with diameter BC pointing inside it. The yellow region is formed by removing the latter semicircle from the union of the former two. The cyan region is just the interior of triangle ABC.)

1.2 Solution

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2 Exercise 2

2.1 Problem

True or False: The pairing in Exercise 1.1.8 is unique (i.e., there is only one way to divide the set $\{1, 2, \ldots, n\}$ into pairs satisfying the conditions of Exercise 1.1.8).

2.2 Solution

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3 Exercise 3

3.1 Problem

(a) Would Exercise 1.1.5 still hold if we replaced "80%" by "50%"?

(b) What if we replaced "80%" by "60%"

(c) What rational numbers r satisfying $0 < r < 1$ have the property that Exercise 1.1.5 still holds if "80%" is replaced by "r" throughout the exercise?

3.2 Solution

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4 Exercise 4

4.1 Problem

Let $n > 3$ be an integer that is not prime. Show that we can find positive integers a, b, c such that $n = ab + bc + ca + 1$.

4.2 Solution

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5 Exercise 5

5.1 Problem

Let k be a positive integer. Let b_k be the number $\underbrace{44\cdots4}_{k \text{ times}}\underbrace{88\cdots8}_{k-1 \text{ times}}9$ (written in base 10). For instance, $b_5 = 4444488889$.

Prove that b_k is a perfect square.

(A *perfect square* means the square of an integer.)

5.2 Solution

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6 Exercise 6

6.1 Problem

Let k be a real number. Consider the sequence $(a_0, a_1, a_2, \ldots)$ of real numbers defined recursively by

$$\begin{aligned} a_0 &= 1; \\ a_n &= kn + (-1)^n a_{n-1} \qquad \text{for all } n \geq 1. \end{aligned}$$

Compute a_{2023}.

6.2 Solution

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7 Exercise 7

7.1 Problem

Find a function $f : \mathbb{Z} \to \mathbb{Z}$ such that each $x \in \mathbb{Z}$ satisfies

$$f(f(x)) = x + 2 \qquad \text{but} \qquad f(x) \neq x + 1.$$

7.2 Solution

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8 Exercise 8

8.1 Problem

Given three nonzero reals a, b, c, solve the system of equations

$$\begin{cases} x^2 = a + (y - z)^2 ; \\ y^2 = b + (z - x)^2 ; \\ z^2 = c + (x - y)^2 \end{cases}$$

in real numbers x, y, z.

8.2 Solution

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9 Exercise 9

9.1 Problem

Let $(a_1, a_2, a_3, \ldots)$ be a sequence of reals such that all positive integers n satisfy

$$1a_1 + 2a_2 + \cdots + na_n = a_{n+1} - 1.$$

Set $c := a_1$. Find an explicit formula for a_n in terms of c.

9.2 Solution

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10 Exercise 10

10.1 Problem

Let I be an interval on $\mathbb{R}$. Let $f : I \to \mathbb{R}$ be a function. Assume that

$$f(a) + f(b) \geq 2f\left(\frac{a+b}{2}\right) \qquad \text{for all } a, b \in I.$$

(A function f satisfying this assumption is said to be *midpoint-convex* on I.)

(a) Prove that

$$f(a) + f(b) + f(c) + f(d) \geq 4f\left(\frac{a+b+c+d}{4}\right) \qquad \text{for all } a, b, c, d \in I.$$

(b) Prove that

$$f(a) + f(b) + f(c) \geq 3f\left(\frac{a+b+c}{3}\right) \qquad \text{for all } a, b, c \in I.$$

[**Hint:** To prove part **(b)**, apply part **(a)** with an appropriate choice of d.]

10.2 Solution

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References