Math 530 Spring 2022, Lecture 9: multigraphs and digraphs

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Multigraphs (cont'd)

1.1. Eulerian circuits and walks

Recall that a walk of a multigraph G is said to be **Eulerian** if each edge of G appears exactly once in this walk.

Last time, we stated the following theorem:

Theorem 1.1.1 (Euler, Hierholzer)**.** Let G be a connected multigraph. Then:

(a) The multigraph G has an Eulerian circuit if and only if each vertex of G has even degree.

(b) The multigraph G has an Eulerian walk if and only if all but at most two vertices of G have even degree.

We have already explained why the "$\Longrightarrow$" directions of both parts **(a)** and **(b)** are true. Today, we will prove the "$\Longleftarrow$" directions.

We prepared for this a bit last time:

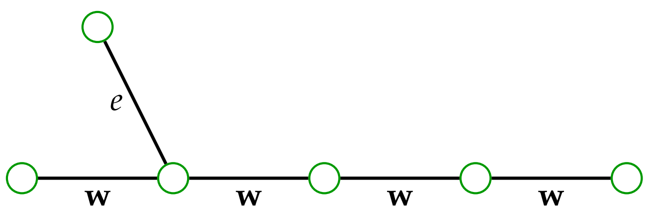
Definition 1.1.2. Let G be a multigraph. A **trail** of G means a walk of G whose edges are distinct.

We showed the following:

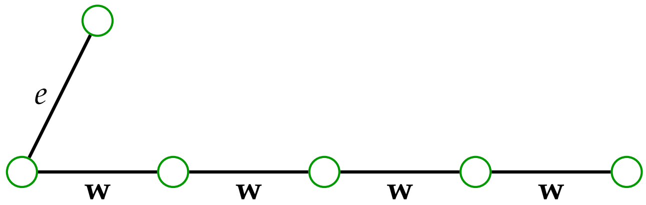
Lemma 1.1.3. Let G be a multigraph with at least one vertex. Then, G has a longest trail.

Our goal now is to show that under appropriate conditions, such a longest trail will be Eulerian. This will require two lemmas.

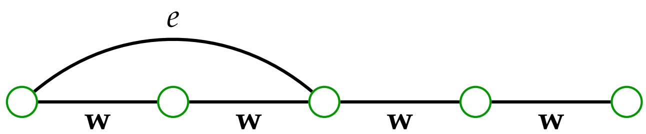
First, one more piece of notation: We say that an edge e of a multigraph G **intersects** a walk $\mathbf{w}$ if at least one endpoint of e is a vertex of $\mathbf{w}$. Here is how this can look like:



(here, the edges of **w** are marked with a "**w**" underneath them) or



(here, the endpoint of e that is a vertex of **w** happens to be the starting point of **w**) or



(here, both endpoints of e happen to be vertices of **w**). Be careful with such pictures, though: A walk doesn't have to be a path; it can visit a vertex any number of times!

Lemma 1.1.4. Let G be a connected multigraph. Let **w** be a walk of G. Assume that there exists an edge of G that is not an edge of **w**.

Then, there exists an edge of G that is not an edge of **w** but intersects **w**.

Proof. We assumed that there exists an edge of G that is not an edge of **w**. Pick such an edge, and call it f.

A "**w**-f-path" will mean a path from a vertex of **w** to an endpoint of f. Such a path clearly exists, since G is connected. Thus, we can pick a **shortest** such path. If this shortest path has length 0, then we are done (since f intersects **w** in this case). If not, we consider the first edge of this path. This first edge cannot be an edge of **w**, because otherwise we could remove it from the path and get an even shorter **w**-f-path. But it clearly intersects **w**. So we have found an edge of G that is not an edge of **w** but intersects **w**. This proves the lemma. $\square$

Lemma 1.1.5. Let G be a multigraph such that each vertex of G has even degree. Let **w** be a longest trail of G. Then, **w** is a closed walk.

Proof. Assume the contrary. Let u be the starting point and v the ending point of **w**. Since we assumed that **w** is not a closed walk, we thus have $u \neq v$.

Consider the edges of **w** that contain v. Such edges are of two kinds: those by which **w** enters v (this means that v comes immediately after this edge in **w**), and those by which **w** leaves v (this means that v comes immediately before this edge in **w**). [1] Except for the very last edge of **w**, each edge of the former kind is immediately followed by an edge of the latter kind; conversely, each edge of the latter kind is immediately preceded by an edge of the former kind

[1] Loops whose only endpoint is v count as both.

(since $\mathbf{w}$ starts at the vertex u, which is distinct from v). Hence, the walk $\mathbf{w}$ has exactly one more edge entering v than it has edges leaving v. Thus, the number of edges of $\mathbf{w}$ that contain v (with loops counting twice) is odd. However, the total number of edges of G that contain v (with loops counting twice) is even (because it is the degree of v, but we assumed that each vertex of G has even degree). So these two numbers are distinct. Thus, there is at least one edge of G that contains v but is not an edge of $\mathbf{w}$.

Fix such an edge and call it f. Now, append f to the trail $\mathbf{w}$ at the end. The result will be a trail (since f is not an edge of $\mathbf{w}$) that is longer than $\mathbf{w}$. But this contradicts the fact that $\mathbf{w}$ is a longest trail. Thus, the lemma is proved. $\square$

We can now finish the proof of the Euler–Hierholzer theorem:

Proof of Theorem 1.1.1. **(a)** $\Longrightarrow$: We did this back in Lecture 8.

$\Longleftarrow$: Assume that each vertex of G has even degree.

By Lemma 1.1.3, we know that G has a longest trail. Fix such a longest trail, and call it $\mathbf{w}$. Then, Lemma 1.1.5 shows that $\mathbf{w}$ is a closed walk.

We claim that $\mathbf{w}$ is Eulerian. Indeed, assume the contrary. Then, there exists an edge of G that is not an edge of $\mathbf{w}$. Hence, Lemma 1.1.4 shows that there exists an edge of G that is not an edge of $\mathbf{w}$ but intersects $\mathbf{w}$. Fix such an edge, and call it f.

Since f intersects $\mathbf{w}$, there exists an endpoint v of f that is a vertex of $\mathbf{w}$. Consider this v. Since $\mathbf{w}$ is a **closed** trail, we can WLOG assume that $\mathbf{w}$ starts and ends at v (since we can otherwise achieve this by rotating[2] $\mathbf{w}$). Then, we can append the edge f to the trail $\mathbf{w}$. This results in a new trail (since f is not an edge of $\mathbf{w}$) that is longer than $\mathbf{w}$. And this contradicts the fact that $\mathbf{w}$ is a longest trail of G.

This contradiction proves that $\mathbf{w}$ is Eulerian. Hence, $\mathbf{w}$ is an Eulerian circuit (since $\mathbf{w}$ is a closed walk). Thus, the "$\Longleftarrow$" direction of Theorem 1.1.1 **(a)** is proven.

(b) $\Longrightarrow$: Already proved in Lecture 8.

$\Longleftarrow$: Assume that all but at most two vertices of G have even degree. We must prove that G has a Eulerian walk.

If each vertex of G has even degree, then this follows from Theorem 1.1.1 **(a)**, since every Eulerian circuit is an Eulerian walk. Thus, we WLOG assume that not each vertex of G has even degree. In other words, the number of vertices of G having odd degree is positive.

[2] **Rotating** a closed walk $(w_0, e_1, w_1, e_2, w_2, \ldots, e_k, w_k)$ means moving its first vertex and its first edge to the end, i.e., replacing the walk by $(w_1, e_2, w_2, e_3, w_3, \ldots, e_k, w_k, e_1, w_1)$. This always results in a closed walk again. For example, if $(1, a, 2, b, 3, c, 1)$ is a closed walk, then we can rotate it to obtain $(2, b, 3, c, 1, a, 2)$; then, rotating it one more time, we obtain $(3, c, 1, a, 2, b, 3)$.

Clearly, by rotating a closed walk several times, we can make it start at any of its vertices. Moreover, if we rotate a closed trail, then we obtain a closed trail.

The handshake lemma for multigraphs (which we proved in Lecture 7) shows that the number of vertices of G having odd degree is even. Furthermore, this number is at most 2 (since all but at most two vertices of G have even degree). So this number is even, positive and at most 2. Thus, this number is 2. In other words, the multigraph G has exactly two vertices having odd degree. Let u and v be these two vertices.

Add a new edge e that has endpoints u and v to the multigraph G (do this even if there already is such an edge![3]). Let G' denote the resulting multigraph. Then, in G', each vertex has even degree (since the newly added edge e has increased the degrees of u and v by 1, thus turning them from odd to even). Moreover, G' is still connected (since G was connected, and the newly added edge e can hardly take that away). Thus, we can apply Theorem 1.1.1 **(a)** to G' instead of G. As a result, we conclude that G' has an Eulerian circuit. Cutting the newly added edge e out of this Eulerian circuit[4], we obtain a Eulerian walk of G. Hence, G has a Eulerian walk. Thus, the "$\Longleftarrow$" direction of Theorem 1.1.1 **(b)** is proven. $\square$

Note: If you look closely at the above proof, you will see hidden in it an algorithm for **finding** Eulerian circuits and walks.[5]

2. Digraphs and multidigraphs

2.1. Definitions

We have so far seen two concepts of graphs: simple graphs and multigraphs.

For all their differences, these two concepts have one thing in common: The two endpoints of an edge are equal in rights. Thus, when defining walks, each edge serves as a "two-way road". Hence, such graphs are good at modelling symmetric relations between things.

We shall now introduce two analogous versions of "graphs" in which the edges have directions. These versions are known as **directed graphs** (short:

[3] This is a time to be grateful for the notion of a multigraph. We could not do this with simple graphs!

[4] More precisely: We rotate this circuit until e becomes its last edge, and then we remove this last edge to obtain a walk.

[5] You might be skeptical about this. After all, in order to apply Lemma 1.1.5, we need a longest trail, so you might wonder how we can find a longest trail to begin with.

Fortunately, we don't need to take Lemma 1.1.5 this literally. Our above proof of Lemma 1.1.5 can be used even if $\mathbf{w}$ is **not** a longest trail. In this case, however, instead of showing that $\mathbf{w}$ is a closed walk, this proof may show us a way how to make $\mathbf{w}$ longer. In other words, by following this proof, we may discover a trail longer than $\mathbf{w}$. In this case, we can replace $\mathbf{w}$ by this longer trail, and then apply Lemma 1.1.5 again. We can repeat this over and over again, until we do end up with a closed walk. (This will eventually happen, since we know that a trail cannot be longer than the total number of edges of G.)

digraphs). In such directed graphs, each edge will have a specified starting point (its "source") and a specified ending point (its "target"). Correspondingly, we will draw these edges as arrows, and we will only allow using them in one direction (viz., from source to target) when we walk down the graph. Here are the definitions in detail:

Definition 2.1.1. A **simple digraph** is a pair (V, A), where V is a finite set, and where A is a subset of $V \times V$.

Definition 2.1.2. Let $D = (V, A)$ be a simple digraph.

(a) The set V is called the **vertex set** of D; it is denoted by $\operatorname{V}(D)$.

Its elements are called the **vertices** (or **nodes**) of D.

(b) The set A is called the **arc set** of D; it is denoted by $\operatorname{A}(D)$.

Its elements are called the **arcs** (or **directed edges**) of D.

When u and v are two elements of V, we will occasionally use uv as a shorthand for the pair (u, v). Note that this means an ordered pair now!

(c) If (u, v) is an arc of D (or, more generally, a pair in $V \times V$), then u is called the **source** of this arc, and v is called the **target** of this arc.

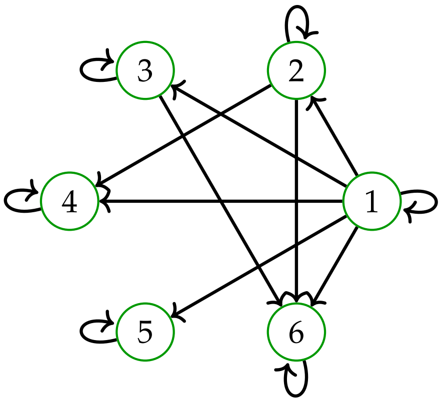
(d) We draw D as follows: We represent each vertex of D by a point, and each arc uv by an arrow that goes from the point representing u to the point representing v.

(e) An arc (u, v) is called a **loop** (or **self-loop**) if $u = v$. (In other words, an arc is a loop if and only if its source is its target.)

Example 2.1.3. For each $n \in \mathbb{N}$, we define the **divisibility digraph on** $\{1, 2, \ldots, n\}$ to be the simple digraph (V, A), where $V = \{1, 2, \ldots, n\}$ and

$$A = \{(i, j) \in V \times V \mid i \text{ divides } j\}.$$

For example, for $n = 6$, this digraph looks as follows:



. (1)

Note that simple digraphs (unlike simple graphs) are allowed to have loops (i.e., arcs of the form (v, v)).

Definition 2.1.4. A **multidigraph** is a triple (V, A, ψ), where V and A are two finite sets, and $\psi : A \to V \times V$ is a map.

Definition 2.1.5. Let $D = (V, A, \psi)$ be a multidigraph.

(a) The set V is called the **vertex set** of D; it is denoted by $\operatorname{V}(D)$.

Its elements are called the **vertices** (or **nodes**) of D.

(b) The set A is called the **arc set** of D; it is denoted by $\operatorname{A}(D)$.

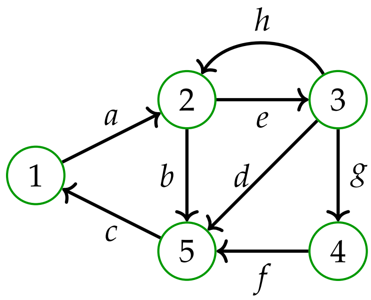
Its elements are called the **arcs** (or **directed edges**) of D.

(c) If a is an arc of D, and if $\psi(a) = (u, v)$, then the vertex u is called the **source** of a, and the vertex v is called the **target** of a.

(d) We draw D as follows: We represent each vertex of D by a point, and each arc a by an arrow that goes from the point representing u to the point representing v, where $(u, v) = \psi(a)$.

Example 2.1.6. Here is a multidigraph:

$D =$



. (2)

Formally speaking, this multidigraph is the triple (V, A, ψ), where $V = \{1, 2, 3, 4, 5\}$ and $A = \{a, b, c, d, e, f, g, h\}$ and $\psi(a) = (1, 2)$ and $\psi(b) = (2, 5)$ and so on.

Thus, simple digraphs and multidigraphs are analogues of simple graphs and multigraphs, respectively, in which the edges have been replaced by arcs ("edges endowed with a direction"). The analogy is perfect but for the fact that simple graphs forbid loops but simple digraphs allow loops (but different authors have different opinions on this).

Convention 2.1.7. The word "**digraph**" means either "simple digraph" or "multidigraph", depending on the context.

The word "digraph" was originally a shorthand for "**directed graph**", but by now it is a technical term that is perfectly understood by everyone in the subject. (It is also understood by linguists, but in a rather different way.)

2.2. Outdegrees and indegrees

What can we do with digraphs? Many of the things we have done with graphs can be modified to work with digraphs (although not all their properties will still hold). For example, the notion of the degree of a vertex in a graph has the following two counterpart notions for digraphs:

Definition 2.2.1. Let D be a digraph with vertex set V. (This can be either a simple digraph or a multidigraph.) Let $v \in V$ be any vertex. Then:

- **(a)** The **outdegree** of v denotes the number of arcs of D whose source is v. This outdegree is denoted $\deg^+ v$.
- **(b)** The **indegree** of v denotes the number of arcs of D whose target is v. This indegree is denoted $\deg^- v$.

Example 2.2.2. In the divisibility digraph on $\{1,2,3,4,5,6\}$ (see (1) for a drawing), we have

$$\begin{array}{llll}
\deg^+ 1 = 6, & \deg^- 1 = 1, & \deg^+ 2 = 3, & \deg^- 2 = 2, \\
\deg^+ 3 = 2, & \deg^- 3 = 2, & \deg^+ 4 = 1, & \deg^- 4 = 3, \\
\deg^+ 5 = 1, & \deg^- 5 = 2, & \deg^+ 6 = 1, & \deg^- 6 = 4.
\end{array}$$

Here is an analogue of the fact that in a graph, the sum of all degrees is twice the number of edges:

Proposition 2.2.3 (diEuler)**.** Let D be a digraph with vertex set V and arc set A. Then,

$$\sum_{v\in V} \deg^+ v = \sum_{v\in V} \deg^- v = |A|.$$

Proof. By the definition of an outdegree, we have

$$\deg^+ v = (\text{the number of arcs of } D \text{ whose source is } v)$$

for each $v \in V$. Thus,

$$\begin{aligned}
\sum_{v\in V} \deg^+ v &= \sum_{v\in V} (\text{the number of arcs of } D \text{ whose source is } v) \\
&= (\text{the number of all arcs of } D) \\
&\qquad \left(\begin{array}{c}\text{since each arc of } D \text{ has exactly one source,} \\ \text{and thus is counted exactly once in the sum}\end{array}\right) \\
&= |A|.
\end{aligned}$$

Similarly, $\sum\limits_{v\in V} \deg^- v = |A|$. $\square$

("diEuler" is not a real mathematician; I just gave that moniker to the above proposition to stress its analogy with Euler's 1736 result.)

2.3. Subdigraphs

Just as we defined subgraphs of a multigraph, we can define subdigraphs (or "submultidigraphs", to be very precise) of a digraph:

Definition 2.3.1. Let $D = (V, A, \psi)$ be a multidigraph.

(a) A **submultidigraph** (or, for short, **subdigraph**) of D means a multidigraph of the form $E = (W, B, \chi)$, where $W \subseteq V$ and $B \subseteq A$ and

$\chi = \psi\mid_B$. In other words, a submultidigraph of D means a multidigraph E whose vertices are vertices of D and whose arcs are arcs of D and whose arcs have the same sources and targets in E as they have in D.

(b) Let S be a subset of V. The **induced subdigraph of** D **on the set** S denotes the subdigraph

$$(S, \ A', \ \psi\mid_{A'})$$

of D, where

$$A' := \{a \in A \ \mid \ \text{both the source and the target of } a \text{ belong to } S\}.$$

In other words, it denotes the subdigraph of D whose vertices are the elements of S, and whose arcs are precisely those arcs of D whose sources and targets both belong to S. We denote this induced subdigraph by $D[S]$.

(c) An **induced subdigraph** of D means a subdigraph of D that is the induced subdigraph of D on S for some $S \subseteq V$.

2.4. Conversions

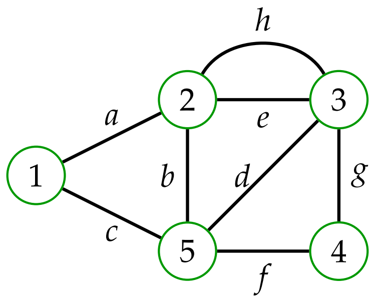
2.4.1. Multidigraphs to multigraphs

Any multidigraph D can be turned into an (undirected) graph G by "removing the arrowheads" (aka "forgetting the directions of the arcs"):

Definition 2.4.1. Let D be a multidigraph. Then, D^{und} will denote the multigraph obtained from D by replacing each arc with an edge whose endpoints are the source and the target of this arc. Formally, this is defined as follows: If $D = (V, A, \psi)$, then $D^{\text{und}} = (V, A, \varphi)$, where the map $\varphi : A \to \mathcal{P}_{1,2}(V)$ sends each arc $a \in A$ to the set of the entries of $\psi(a)$ (that is, to the set consisting of the source of a and the target of a).

For example, if D is the multidigraph from (2), then D^{und} is the following multigraph:

$D^{\text{und}} =$



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Conversely, any multigraph can be turned into a multidigraph by “duplicating” each edge and making it go in both directions. More about this next time.