Math 530 Spring 2022, Lecture 6: Hamiltonian paths

website: `https://www.cip.ifi.lmu.de/~grinberg/t/22s`

1. Simple graphs (cont'd)

1.1. Dominating sets (cont'd)

Last time, we have defined the notion of dominating sets and almost proved the following theorem:

Theorem 1.1.1 (Brouwer's dominating set theorem)**.** Let G be a simple graph. Then, the number of dominating sets of G is odd.

For convenience, let me repeat the almost-proof:

Proof of Brouwer's dominating set theorem, attempt 1. Write G as (V, E). Recall that $\mathcal{P}(V)$ denotes the set of all subsets of V.

Construct a new graph H with the vertex set $\mathcal{P}(V)$ as follows: Two subsets A and B of V are adjacent as vertices of H if and only if (A, B) is a detached pair[1]. (Note that if the original graph G has n vertices, then this graph H has 2^n vertices. It is huge!)

I claim that the vertices of H that have odd degree are precisely the subsets of V that are dominating. In other words:

> *Claim 1:* Let A be a subset of V. Then, the vertex A of H has odd degree if and only if A is a dominating set of G.

[*Proof of Claim 1:* We let $N(A)$ denote the set of all vertices of G that have a neighbor in A. (This may or may not be disjoint from A.)

The neighbors of A (as a vertex in H) are precisely the subsets B of V such that (A, B) is a detached pair (by the definition of H). In other words, they are the subsets B of V that are disjoint from A and also have no neighbors in A (by the definition of a "detached pair"). In other words, they are the subsets B of V that are disjoint from A and also disjoint from $N(A)$. In other words, they are the subsets of the set $V \setminus (A \cup N(A))$. Hence, the number of such subsets B is $2^{|V \setminus (A \cup N(A))|}$.

The degree of A (as a vertex of H) is the number of neighbors of A in H. Thus, this degree is $2^{|V \setminus (A \cup N(A))|}$ (because we have just shown that the number of neighbors of A is $2^{|V \setminus (A \cup N(A))|}$). But 2^k is odd if and only if $k = 0$.

[1] Recall that a **detached pair** means a pair (A, B) of two disjoint subsets A and B of V such that there exists no edge $ab \in E$ with $a \in A$ and $b \in B$.

Thus, we conclude that the degree of A (as a vertex of H) is odd if and only if $|V \setminus (A \cup N(A))| = 0$. The condition $|V \setminus (A \cup N(A))| = 0$ can be rewritten as follows:

$$\begin{aligned}
& (|V \setminus (A \cup N(A))| = 0) \\
\iff & (V \setminus (A \cup N(A)) = \varnothing) \\
\iff & (V \subseteq A \cup N(A)) \\
\iff & (V \setminus A \subseteq N(A)) \\
\iff & (\text{each vertex } v \in V \setminus A \text{ belongs to } N(A)) \\
\iff & (\text{each vertex } v \in V \setminus A \text{ has a neighbor in } A) \\
\iff & (A \text{ is dominating}) \qquad (\text{by the definition of “dominating”}).
\end{aligned}$$

Thus, what we have just shown is that the degree of A (as a vertex of H) is odd if and only if A is dominating. This proves Claim 1.]

Claim 1 shows that the vertices of H that have odd degree are precisely the dominating sets of G. But the handshake lemma (the first corollary in Lecture 2) tells us that any simple graph has an even number of vertices of odd degree. Applying this to H, we conclude that there is an even number of dominating sets of G.

Huh? We want to show that there is an **odd** number of dominating sets of G, not an even number! Why did we just get the opposite result? □

So what was the mistake in our reasoning?

The mistake is that our definition of H requires the vertex $\varnothing$ of H to be adjacent to itself (since $(\varnothing, \varnothing)$ is a detached pair); but a vertex of a simple graph cannot be adjacent to itself. So we need to tweak the definition of H somewhat:

Correction of the above proof. Define the graph H as above, but do not try to have $\varnothing$ adjacent to itself. (This is the only vertex that creates any trouble, because a detached pair (A, B) cannot satisfy $A = B$ unless both A and B are $\varnothing$.)

We WLOG assume that $V \neq \varnothing$ (otherwise, the claim is obvious). Thus, the empty set $\varnothing$ is not dominating.

Our Claim 1 needs to be modified as follows:

> *Claim 1′:* Let A be a subset of V. Then, the vertex A of H has odd degree if and only if A is empty or a dominating set of G.

This can be proved in the same way as we "proved" Claim 1 above; we just need to treat the $A = \varnothing$ case separately now (but this case is easy: $\varnothing$ is adjacent to all other vertices of H, and thus has degree $2^{|V|} - 1$, which is odd).

So we conclude (using the handshake lemma) that the number of empty or dominating sets is even. Subtracting 1 for the empty set, we conclude that the number of dominating sets is odd (since the empty set is not dominating). This proves Brouwer's theorem. □

There are other ways to prove Brouwer's theorem as well. A particularly nice one was found by Irene Heinrich and Peter Tittmann in 2017; they gave an "explicit" formula for the number of dominating sets that shows that this number is odd ([HeiTit17, Theorem 8], restated using the language of detached pairs):

Theorem 1.1.2 (Heinrich–Tittmann formula)**.** Let $G = (V, E)$ be a simple graph with n vertices. Assume that $n > 0$.

Let α be the number of all detached pairs (A, B) such that both numbers $|A|$ and $|B|$ are even and positive.

Let β be the number of all detached pairs (A, B) such that both numbers $|A|$ and $|B|$ are odd.

Then:

(a) The numbers α and β are even.

(b) The number of dominating sets of G is $2^n - 1 + \alpha - \beta$.

Part **(a)** of this theorem is obvious (recall that if (A, B) is a detached pair, then so is (B, A)). Part **(b)** is the interesting part. In [17s, §3.3–§3.4], I give a long but elementary proof.

More recently ([HeiTit18]), Heinrich and Tittmann have refined their formula to allow counting dominating sets of a given size. I have posed their main result as exercise 5 on homework set #2.

1.2. Hamiltonian paths and cycles

1.2.1. Basics

Now to something different. Here is a quick question: Given a simple graph G, when is there a closed **walk** that contains each vertex of G ?

The answer is easy: When G is connected. Indeed, if a simple graph G is connected, then we can label its vertices by $v_1, v_2, \ldots, v_n$ arbitrarily, and we then get a closed walk by composing a walk from v_1 to v_2 with a walk from v_2 to v_3 with a walk from v_3 to v_4 and so on, ending with a walk from v_n to v_1. This closed walk will certainly contain each vertex. Conversely, such a walk cannot exist if G is not connected.

The question becomes a lot more interesting if we replace "closed walk" by "path" or "cycle". The resulting objects have a name:

Definition 1.2.1. Let $G = (V, E)$ be a simple graph.

(a) A **Hamiltonian path** in G means a walk of G that contains each vertex of G exactly once. Obviously, it is a path.

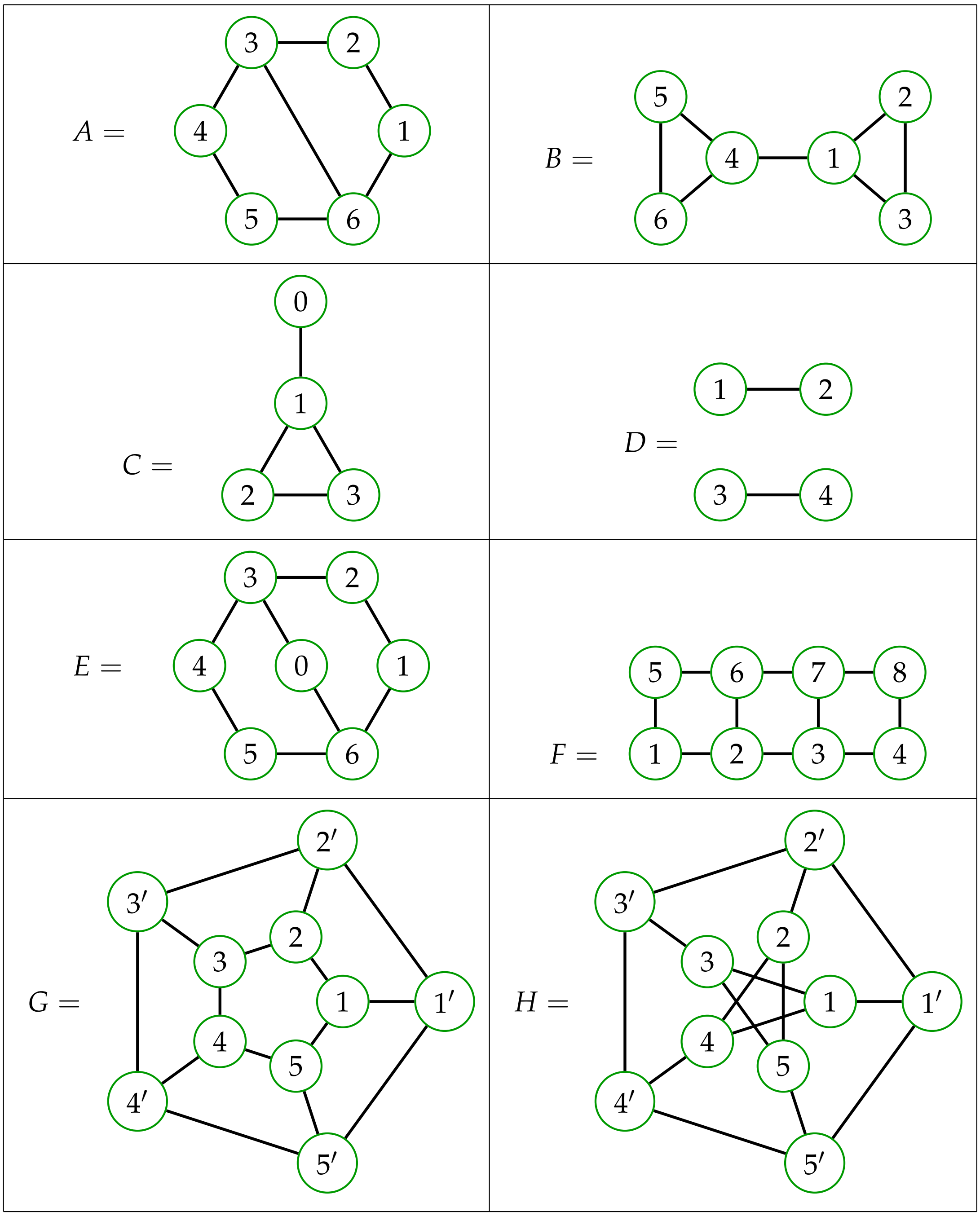
(b) A **Hamiltonian cycle** in G means a cycle $(v_0, v_1, \ldots, v_k)$ of G such that each vertex of G appears exactly once among $v_0, v_1, \ldots, v_{k-1}$.

Some graphs have Hamiltonian paths; some don't. Having a Hamiltonian cycle is even stronger than having a Hamiltonian path, because if $(v_0, v_1, \ldots, v_k)$ is a Hamiltonian cycle of G, then $(v_0, v_1, \ldots, v_{k-1})$ is a Hamiltonian path of G.

Convention 1.2.2. In the following, we will abbreviate:

- "Hamiltonian path" as "**hamp**";
- "Hamiltonian cycle" as "**hamc**".

Example 1.2.3. Which of the following eight graphs have hamps? Which have hamcs?



Answers:

- The graph A has a hamc $(1, 2, 3, 4, 5, 6, 1)$, and thus a hamp $(1, 2, 3, 4, 5, 6)$. (Recall that a graph that has a hamc always has a hamp, since we can simply remove the last vertex from a hamc to obtain a hamp.)

- The graph B has a hamp $(2,3,1,4,5,6)$, but no hamc. The easiest way to see that B has no hamc is the following: The edge 14 is a cut-edge (i.e., removing it renders the graph disconnected), thus a bridge (i.e., an edge that appears in no cycle); therefore, any cycle must stay entirely "on one side" of this edge.

- The graph C has a hamp $(0,1,2,3)$, but no hamc. The argument for the non-existence of a hamc is the same as for B: The edge 01 is a bridge.

- The graph D has neither a hamp nor a hamc, because it is not connected. Only a connected graph can have a hamp.

- The graph E has a hamp $(0,3,2,1,6,5,4)$, but no hamc (checking this requires some work, though).

- The graph F has a hamc $(1,2,3,4,8,7,6,5,1)$, thus also a hamp.

- The graph G has a hamc $(1,2,3,4,5,5',4',3',2',1',1)$, thus also a hamp.

- The graph H (which, by the way, is isomorphic to the Petersen graph) has a hamp $(1,3,5,2,4,4',3',2',1',5')$, but no hamc (but this is not obvious! see the Wikipedia article for an argument).

In general, finding a hamp or a hamc, or proving that none exists, is a hard problem. It can always be solved by brute force (i.e., by trying all lists of distinct vertices and checking if there is a hamp among them, and likewise for hamcs), but this quickly becomes forbiddingly laborious as the size of the graph increases. Some faster algorithms exist (in particular, there is one of running time $O\left(n^2 2^n\right)$, where n is the number of vertices), but no polynomial-time algorithm is known. The problem (both in its hamp version and in its hamc version) is known to be NP-hard (in the language of complexity theory). In practice, hamps and hamcs can often be found with some wit and perseverance; proofs of their non-existence can often be obtained with some logic and case analysis (see the above example for some sample arguments). See the Wikipedia page for "Hamiltonian path problem" for more information.

The problem of finding hamps is related to the so-called "traveling salesman problem" (TSP), which asks for a hamp with "minimum weight" in a weighted graph (each edge has a number assigned to it, which is called its "weight", and the weight of a hamp is the sum of the weights of the edges it uses). There is a lot of computer-science literature about this problem.

1.2.2. Sufficient criteria: Ore and Dirac

We shall now show some necessary criteria and some sufficient criteria (but no necessary-and-sufficient criteria) for the existence of hamps and hamcs. Here is the most famous sufficient criterion:

Theorem 1.2.4 (Ore)**.** Let $G = (V, E)$ be a simple graph with n vertices, where $n \geq 3$.

Assume that $\deg x + \deg y \geq n$ for any two non-adjacent distinct vertices x and y.

Then, G has a hamc.

There are various proofs of this theorem scattered around; see [Harju14, Theorem 3.6] or [Guicha16, Theorem 5.3.2]. We shall give another proof (following the "Algorithm" section on the Wikipedia page for "Ore's theorem"):

Proof of Theorem 1.2.4. A **listing** (of V) shall mean a list of elements of V that contains each element exactly once. It must clearly be an n-tuple.

The **hamness** of a listing $(v_1, v_2, \ldots, v_n)$ will mean the number of all $i \in \{1, 2, \ldots, n\}$ such that $v_i v_{i+1} \in E$. Here, we set $v_{n+1} = v_1$. (Visually, it is best to represent a listing $(v_1, v_2, \ldots, v_n)$ by drawing the vertices $v_1, v_2, \ldots, v_n$ on a circle in this order. Its hamness then counts how often two successive vertices on the circle are adjacent in the graph G.) Note that the hamness of a listing $(v_1, v_2, \ldots, v_n)$ does not change if we cyclically rotate the listing (i.e., transform it into $(v_2, v_3, \ldots, v_n, v_1)$).

Clearly, if we can find a listing $(v_1, v_2, \ldots, v_n)$ of hamness $\geq n$, then all of v_1v_2, v_2v_3, $\ldots$, v_nv_1 are edges of G, and thus $(v_1, v_2, \ldots, v_n, v_1)$ is a hamc of G. Thus, we need to find a listing of hamness $\geq n$.

To do so, I will show that if you have a listing of hamness $< n$, then you can slightly modify it to get a listing of larger hamness. In other words, I will show the following:

> *Claim 1:* Let $(v_1, v_2, \ldots, v_n)$ be a listing of hamness $k < n$. Then, there exists a listing of hamness larger than k.

[*Proof of Claim 1:* Since the listing $(v_1, v_2, \ldots, v_n)$ has hamness $k < n$, there exists some $i \in \{1, 2, \ldots, n\}$ such that $v_i v_{i+1} \notin E$. Pick such an i. Thus, the vertices v_i and v_{i+1} of G are non-adjacent (and distinct). The "$\deg x + \deg y \geq n$" assumption of the theorem thus yields $\deg(v_i) + \deg(v_{i+1}) \geq n$.

However,

$$\begin{aligned}
\deg(v_i) &= |\{w \in V \mid v_i w \in E\}| \\
&= \left|\{j \in \{1, 2, \ldots, n\} \mid v_i v_j \in E\}\right| \\
&= \left|\{j \in \{1, 2, \ldots, n\} \setminus \{i\} \mid v_i v_j \in E\}\right|
\end{aligned}$$

(because $j = i$ could not satisfy $v_i v_j \in E$ anyway) and

$$\begin{aligned} \deg(v_{i+1}) &= |\{w \in V \mid v_{i+1}w \in E\}| \\ &= \left|\left\{j \in \{1,2,\ldots,n\} \mid v_{i+1}v_{j+1} \in E\right\}\right| \\ &\qquad \left(\begin{array}{c}\text{since } (v_2, v_3, \ldots, v_{n+1}) \text{ is a listing of } V \\ (\text{because } v_{n+1} = v_1)\end{array}\right) \\ &= \left|\left\{j \in \{1,2,\ldots,n\} \setminus \{i\} \mid v_{i+1}v_{j+1} \in E\right\}\right| \end{aligned}$$

(because $j = i$ could not satisfy $v_{i+1}v_{j+1} \in E$ anyway). In light of these two equalities, we can rewrite the inequality $\deg(v_i) + \deg(v_{i+1}) \geq n$ as

$$\begin{aligned} &\left|\left\{j \in \{1,2,\ldots,n\} \setminus \{i\} \mid v_i v_j \in E\right\}\right| \\ &\qquad + \left|\left\{j \in \{1,2,\ldots,n\} \setminus \{i\} \mid v_{i+1}v_{j+1} \in E\right\}\right| \geq n. \end{aligned}$$

Thus, the two subsets $\left\{j \in \{1,2,\ldots,n\} \setminus \{i\} \mid v_i v_j \in E\right\}$ and $\left\{j \in \{1,2,\ldots,n\} \setminus \{i\} \mid v_{i+1}v_{j+1} \in E\right\}$ of the $(n-1)$-element set $\{1,2,\ldots,n\} \setminus \{i\}$ have total size $\geq n$ (that is, the sum of their sizes is $\geq n$). Hence, these two subsets must overlap (i.e., have an element in common). In other words, there exists a $j \in \{1,2,\ldots,n\} \setminus \{i\}$ that satisfies both $v_i v_j \in E$ and $v_{i+1}v_{j+1} \in E$. Pick such a j.

Now, consider a new listing obtained from the old listing $(v_1, v_2, \ldots, v_n)$ as follows:

- First, cyclically rotate the old listing so that it begins with v_{i+1}. Thus, you get the listing $(v_{i+1}, v_{i+2}, \ldots, v_n, v_1, v_2, \ldots, v_i)$.
- Then, reverse the part of the listing starting at v_{i+1} and ending at v_j. Thus, you get the new listing

$$\left(\underbrace{v_j, v_{j-1}, \ldots, v_{i+1}}_{\substack{\text{This is the reversed part;} \\ \text{it may or may not “wrap around”} \\ \text{(i.e., contain } \ldots, v_1, v_n, \ldots \text{ somewhere).}}}, \underbrace{v_{j+1}, v_{j+2}, \ldots, v_i}_{\substack{\text{This is the part that} \\ \text{was not reversed.}}}\right).$$

This is the new listing we want.

I claim that this new listing has hamness larger than k. Indeed, rotating the old listing clearly did not change its hamness. But reversing the part from v_{i+1} to v_j clearly did: After the reversal, the edges $v_i v_{i+1}$ and $v_j v_{j+1}$ no longer count towards the hamness (if they were edges to begin with), but the edges $v_i v_j$ and $v_{i+1}v_{j+1}$ started counting towards the hamness. This is a good bargain, because it means that the hamness gained $+2$ from the newly-counted edges $v_i v_j$ and

$v_{i+1}v_{j+1}$ (which, as we know, both exist), while only losing 0 or 1 (since the edge v_iv_{i+1} did not exist, whereas the edge v_jv_{j+1} may or may not have been lost). Thus, the hamness of the new listing is larger than the hamness of the old listing either by 1 or 2. In other words, it is larger than m by at least 1 or 2. This proves Claim 1.]

Now, we can start with **any** listing of V and keep modifying it using Claim 1, increasing its hamness each time, until its hamness becomes $\geq n$. But once its hamness is $\geq n$, we have found a hamc (as explained above). Theorem 1.2.4 is thus proven. □

Corollary 1.2.5 (Dirac)**.** Let $G = (V, E)$ be a simple graph with n vertices, where $n \geq 3$.

Assume that $\deg x \geq \frac{n}{2}$ for each vertex $x \in V$.

Then, G has a hamc.

Proof. Follows from Ore's theorem, since any two vertices x and y of G satisfy $\underbrace{\deg x}_{\geq \frac{n}{2}} + \underbrace{\deg y}_{\geq \frac{n}{2}} \geq \frac{n}{2} + \frac{n}{2} = n$. □

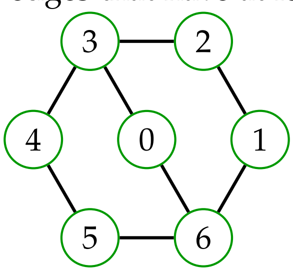
1.2.3. A necessary criterion

So much for sufficient criteria. What about necessary criteria?

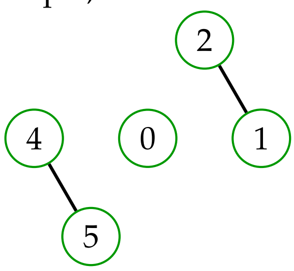
Proposition 1.2.6. Let $G = (V, E)$ be a simple graph.

For each subset S of V, we let $G \setminus S$ be the induced subgraph of G on the set $V \setminus S$. (In other words, this is the graph obtained from G by removing all vertices in S and removing all edges that have at least one endpoint in S.)

(For example, if $G =$



and $S = \{3, 6\}$, then $G \setminus S =$



.)

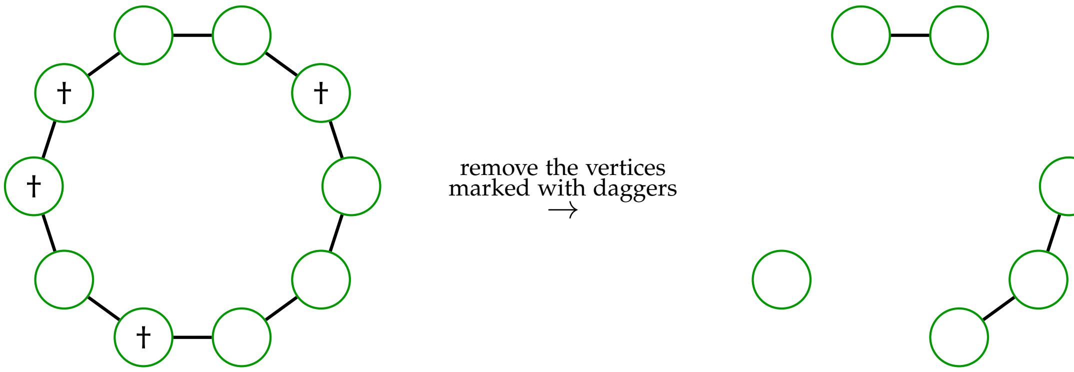
Also, we let $b_0(H)$ denote the number of connected components of a simple graph H.

(a) If G has a hamc, then every nonempty $S \subseteq V$ satisfies $b_0(G \setminus S) \leq |S|$.

(b) If G has a hamp, then every $S \subseteq V$ satisfies $b_0(G \setminus S) \leq |S| + 1$.

For example, part **(a)** of this proposition shows that the graph E from Example 1.2.3 has no hamc, because if we take S to be $\{3, 6\}$, then $b_0(G \setminus S) = 3$ whereas $|S| = 2$. Thus, the proposition can be used to rule out the existence of hamps and hamcs in some cases.

Proof of Proposition 1.2.6. **(a)** Let $S \subseteq V$ be a nonempty set. If we cut $|S|$ many vertices out of a cycle, then the cycle splits into at most $|S|$ paths:



Of course, our graph G itself may not be a cycle, but if it has a hamc, then the removal of the vertices in S will split the hamc into at most $|S|$ paths (according to the preceding sentence), and thus the graph $G \setminus S$ will have $\leq |S|$ many components (just using the surviving edges of the hamc alone). Taking into account all the other edges of G can only decrease the number of components.

(b) This is analogous to part **(a)**. □

This proposition often (but not always) gives a quick way of convincing yourself that a graph has no hamc or hamp. Alas, its converse is false. Case in point: The Petersen graph has no hamc, but it does satisfy the "every nonempty $S \subseteq V$ satisfies $b_0(G \setminus S) \leq |S|$" condition of Proposition 1.2.6 **(a)**.

1.2.4. Hypercubes

Now, let us move on to a concrete example of a graph that has a hamc.

Definition 1.2.7. Let $n \in \mathbb{N}$. The ***n*-hypercube** Q_n (more precisely, the ***n*-th hypercube graph**) is the simple graph with vertex set

$$\{0,1\}^n = \{(a_1, a_2, \ldots, a_n) \mid \text{each } a_i \text{ belongs to } \{0,1\}\}$$

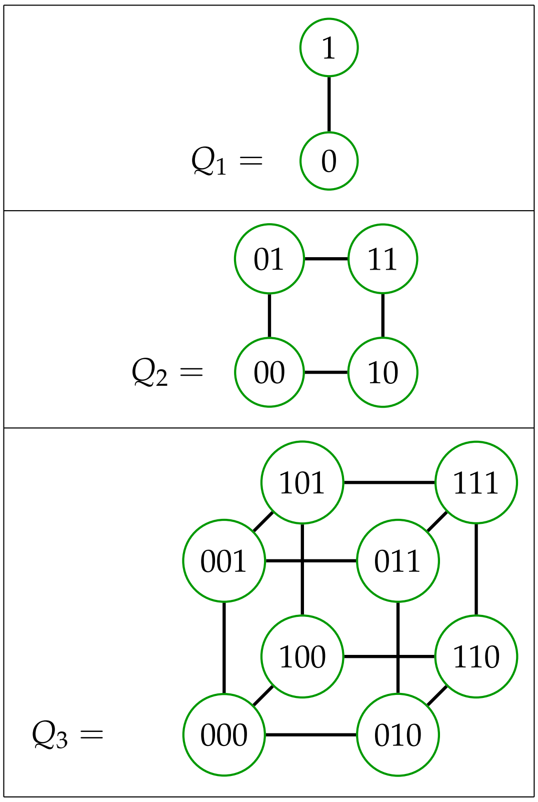
and edge set defined as follows: A vertex $(a_1, a_2, \ldots, a_n) \in \{0,1\}^n$ is adjacent to a vertex $(b_1, b_2, \ldots, b_n) \in \{0,1\}^n$ if and only if there exists **exactly** one

$i \in \{1, 2, \ldots, n\}$ such that $a_i \neq b_i$. (For example, in Q_4, the vertex $(0, 1, 1, 0)$ is adjacent to $(0, 1, 0, 0)$.)

The elements of $\{0, 1\}^n$ are often called **bitstrings** (or **binary words**), and their entries are called their **bits** (or **letters**). So two bitstrings are adjacent in Q_n if and only if they differ in exactly one bit.

We often write a bitstring $(a_1, a_2, \ldots, a_n)$ as $a_1 a_2 \cdots a_n$. (For example, we write $(0, 1, 1, 0)$ as 0110.)

Example 1.2.8. Here is how the n-hypercubes Q_n look like for $n = 1, 2, 3$:



This should explain the name "hypercube". The 0-hypercube Q_0 is a graph with just one vertex (namely, the empty bitstring $()$).

Theorem 1.2.9 (Gray)**.** Let $n \geq 2$. Then, the graph Q_n has a hamc.

Such hamcs are known as **Gray codes**. They are circular lists of bitstrings of length n such that two consecutive bitstrings in the list always differ in exactly one bit. See the Wikipedia article on "Gray codes" for applications.

Proof of Theorem 1.2.9. We will show something stronger:

> *Claim 1:* For each $n \geq 1$, the n-hypercube Q_n has a hamp from $00 \cdots 0$ to $100 \cdots 0$.
>
> (Keep in mind that $00 \cdots 0$ and $100 \cdots 0$ are bitstrings, not numbers:
>
> $$00 \cdots 0 = \Bigg(\underbrace{0, 0, \ldots, 0}_{n \text{ zeroes}} \Bigg); \qquad 100 \cdots 0 = \Bigg(1, \underbrace{0, 0, \ldots, 0}_{n-1 \text{ zeroes}} \Bigg).$$
>
>)

[*Proof of Claim 1:* We induct on n.

Induction base: A look at Q_1 reveals a hamp from 0 to 1.

Induction step: Fix $N \geq 2$. We assume that Claim 1 holds for $n = N - 1$. In other words, Q_{N-1} has a hamp from $\underbrace{00\cdots0}_{N-1 \text{ zeroes}}$ to $1\underbrace{00\cdots0}_{N-2 \text{ zeroes}}$. Let $\mathbf{p}$ be such a hamp.

By attaching a 0 to the front of each bitstring (= vertex) in $\mathbf{p}$, we obtain a path $\mathbf{q}$ from $\underbrace{00\cdots0}_{N \text{ zeroes}}$ to $01\underbrace{00\cdots0}_{N-2 \text{ zeroes}}$ in Q_N.

By attaching a 1 to the front of each bitstring (= vertex) in $\mathbf{p}$, we obtain a path $\mathbf{r}$ from $1\underbrace{00\cdots0}_{N-1 \text{ zeroes}}$ to $11\underbrace{00\cdots0}_{N-2 \text{ zeroes}}$ in Q_N.

Now, we assemble a hamp from $\underbrace{00\cdots0}_{N \text{ zeroes}}$ to $1\underbrace{00\cdots0}_{N-1 \text{ zeroes}}$ in Q_N as follows:

- Start at $\underbrace{00\cdots0}_{N \text{ zeroes}}$, and follow the path $\mathbf{q}$ to its end (i.e., to $01\underbrace{00\cdots0}_{N-2 \text{ zeroes}}$).
- Then, move to the adjacent vertex $11\underbrace{00\cdots0}_{N-2 \text{ zeroes}}$.
- Then, follow the path $\mathbf{r}$ backwards, ending up at $1\underbrace{00\cdots0}_{N-1 \text{ zeroes}}$.

This shows that Claim 1 holds for $n = N$, too.]

Claim 1 tells us that the n-hypercube Q_n has a hamp from $00\cdots0$ to $100\cdots0$. Since its starting point $00\cdots0$ and its ending point $100\cdots0$ are adjacent, we can turn this hamp into a hamc by appending the starting point $00\cdots0$ again at the end. This proves Theorem 1.2.9. □

1.2.5. Cartesian products

Theorem 1.2.9 can in fact be generalized. To state the generalization, we define the **Cartesian product** of two graphs:

Definition 1.2.10. Let $G = (V, E)$ and $H = (W, F)$ be two simple graphs. The **Cartesian product** $G \times H$ of these two graphs is defined to be the graph $(V \times W,\ E' \cup F')$, where

$$E' := \{(v_1, w)(v_2, w) \mid v_1v_2 \in E \text{ and } w \in W\} \qquad \text{and}$$
$$F' := \{(v, w_1)(v, w_2) \mid w_1w_2 \in F \text{ and } v \in V\}.$$

In other words, it is the graph whose vertices are pairs $(v, w) \in V \times W$ consisting of a vertex of G and a vertex of H, and whose edges are of the forms

$$(v_1, w)(v_2, w) \qquad \text{where } v_1v_2 \in E \text{ and } w \in W$$

and

$$(v, w_1)(v, w_2) \qquad \text{where } w_1 w_2 \in F \text{ and } v \in V.$$

For example, the Cartesian product $G \times P_2$ of a simple graph G with the 2-path graph P_2 can be constructed by overlaying two copies of G and additionally joining each vertex of the first copy to the corresponding vertex of the second copy by an edge. (The vertices of the first copy are the $(v, 1)$, whereas the vertices of the second copy are the $(v, 2)$.) In particular, it is easy to see the following:

Proposition 1.2.11. We have $Q_n \cong Q_{n-1} \times P_2$ for each $n \geq 1$.

Now, we claim the following:

Theorem 1.2.12. Let G and H be two simple graphs. Assume that each of the two graphs G and H has a hamp. Then:

(a) The Cartesian product $G \times H$ has a hamp.

(b) Now assume furthermore that at least one of the two numbers $|V(G)|$ and $|V(H)|$ is even, and that both numbers $|V(G)|$ and $|V(H)|$ are larger than 1. Then, the Cartesian product $G \times H$ has a hamc.

Proof. See the solution to Exercise 1 on homework set #2 from Spring 2017. □

Now, Theorem 1.2.9 can be reproved (again by inducting on n) using Theorem 1.2.12 **(b)** and Proposition 1.2.11, since P_2 has a hamp and since $|V(P_2)| = 2$ is even. (Convince yourself that this works!)

1.2.6. Subset graphs

The n-hypercube Q_n can be reinterpreted in terms of subsets of $\{1, 2, \ldots, n\}$. Namely: Let $n \in \mathbb{N}$. Let G_n be the simple graph whose vertex set is the powerset $\mathcal{P}(\{1, 2, \ldots, n\})$ of $\{1, 2, \ldots, n\}$ (that is, the vertices are all 2^n subsets of $\{1, 2, \ldots, n\}$), and whose edges are determined as follows: Two vertices S and T are adjacent if and only if one of the two sets S and T is obtained from the other by inserting an extra element (i.e., we have either $S = T \cup \{s\}$ for some $s \notin T$, or $T = S \cup \{t\}$ for some $t \notin S$). Then, $G_n \cong Q_n$, since the map

$$\begin{aligned} \{0,1\}^n &\to \mathcal{P}(\{1, 2, \ldots, n\}), \\ (a_1, a_2, \ldots, a_n) &\mapsto \{i \in \{1, 2, \ldots, n\} \mid a_i = 1\} \end{aligned}$$

is a graph isomorphism from Q_n to G_n.

Thus, Theorem 1.2.9 shows that for each $n \geq 2$, the graph G_n has a hamc. In other words, for each $n \geq 2$, we can list all subsets of $\{1, 2, \ldots, n\}$ in a circular list in such a way that each subset on this list is obtained from the previous one by inserting or removing a single element. For example, for $n = 3$, here is such a list:

$$\varnothing, \ \{1\}, \ \{1,2\}, \ \{2\}, \ \{2,3\}, \ \{1,2,3\}, \ \{1,3\}, \ \{3\}.$$

A long-standing question only resolved a few years ago asked whether the same can be done with the subsets of $\{1,2,\ldots,n\}$ having size $\dfrac{n\pm 1}{2}$ when n is odd. For example, for $n = 3$, we can do it as follows:

$$\{1\}, \ \{1,2\}, \ \{2\}, \ \{2,3\}, \ \{3\}, \ \{1,3\}.$$

In other words, if $n \geq 3$ is odd, and if G'_n is the induced subgraph of G_n on the set of all subsets J of $\{1,2,\ldots,n\}$ that satisfy $|J| \in \left\{\dfrac{n-1}{2}, \dfrac{n+1}{2}\right\}$, then does G'_n have a hamc?

Since $G_n \cong Q_n$, we can restate this question equivalently as follows: If $n \geq 3$ is odd, and if Q'_n is the induced subgraph of Q_n on the set

$$\left\{ a_1 a_2 \cdots a_n \in \{0,1\}^n \ \mid \ \sum_{i=1}^{n} a_i \in \left\{\frac{n-1}{2}, \frac{n+1}{2}\right\} \right\},$$

then does Q'_n have a hamc?

In 2014, Torsten Mütze proved that the answer is "yes". See [Mutze14] for his truly nontrivial proof, and [Mutze22] for a recent survey of similar questions. (Cf. also change ringing.)

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